Annali di Matematica (2016) 195:623–658
DOI 10.1007/s10231-015-0481-3

The boundary Harnack inequality for variable exponent p-Laplacian, Carleson estimates, barrier functions and $p(\cdot)$-harmonic measures

Tomasz Adamowicz · **Niklas L. P. Lundström**

Received: 13 June 2014 / Accepted: 28 January 2015 / Published online: 20 February 2015

Abstract We investigate various boundary decay estimates for $p(\cdot)$-harmonic functions. For domains in $\mathbb{R}^n$, $n \geq 2$ satisfying the ball condition ($C^{1,1}$-domains), we show the boundary Harnack inequality for $p(\cdot)$-harmonic functions under the assumption that the variable exponent p is a bounded Lipschitz function. The proof involves barrier functions and chaining arguments. Moreover, we prove a Carleson-type estimate for $p(\cdot)$-harmonic functions in NTA domains in $\mathbb{R}^n$ and provide lower and upper growth estimates and a doubling property for a $p(\cdot)$-harmonic measure.

Keywords Ball condition · Boundary Harnack inequality · Harmonic measure · NTA domain · Nonstandard growth equation · p-harmonic

Mathematics Subject Classification Primary 31B52; Secondary 35J92 · 35B09 · 31B25

1 Introduction

The studies of boundary Harnack inequalities for solutions of differential equations have a long history. In the setting of harmonic functions on Lipschitz domains, such a result was first proposed by Kemper [41] and later studied by Ancona [11], Dahlberg [23] and Wu [60]. Subsequently, Kemper's result was extended by Caffarelli et al. [21] to a class of elliptic equations, by Jerison and Kenig [40] to the setting of nontangentially accessible

---

T. Adamowicz was supported by a Grant of National Science Center, Poland (NCN), UMO-2013/09/D/ST1/03681.

T. Adamowicz (✉)
Institute of Mathematics of the Polish Academy of Sciences, 00-656 Warsaw, Poland
e-mail: T.Adamowicz@impan.pl

N. L. P. Lundström
Department of Mathematics and Mathematical Statistics, Umeå University, 90187 Umeå, Sweden
e-mail: niklas.lundstrom@math.umu.se

(NTA) domains, Bañuelos et al. [14] and Bass and Burdzy [15] studied the case of Hölder domains while Aikawa [6] the case of uniform domains. The extension of these results to the more general setting of p-harmonic operators turned out to be difficult, largely due to the nonlinearity of p-harmonic functions for $p \neq 2$. However, recently, there has been a substantial progress in studies of boundary Harnack inequalities for nonlinear Laplacians: Aikawa et al. [7] studied the case of p-harmonic functions in $C^{1,1}$-domains, while in the same time, Lewis and Nyström [45,47,48] began to develop a theory applicable in more general geometries such as Lipschitz and Reifenberg-flat domains. Lewis–Nyström results have been partially generalized to operators with variable coefficients, Avelin et al. [12], Avelin and Nyström [13], and to p-harmonic functions in the Heisenberg group, Nyström [55]. Moreover, in [52], the second author proved a boundary Harnack inequality for p-harmonic functions with $n < p \leq \infty$ vanishing on a m-dimensional hyperplane in $\mathbb{R}^n$ for $0 \leq m \leq n-1$. We also refer to Bhattacharya [18] and Lundström and Nyström [53] for the case $p = \infty$, where the latter investigated A-harmonic and Aronsson-type equations in planar uniform domains. Concerning the applications of boundary Harnack inequalities, we mention free boundary problems and studies of the Martin boundary.

Another recently developing branch of nonlinear analysis is the area of differential equations with nonstandard growth (variable exponent analysis) and related variational functionals. The following equation, called the $p(\cdot)$-Laplace equation, serves as the model example:

$$\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) = 0, \tag{1.1}$$

for a measurable function $p : \Omega \to [1, \infty]$ called a variable exponent. The variational origin of this equation naturally implies that solutions belong to the appropriate Musielak–Orlicz space $W^{1,p(\cdot)}(\Omega)$ (see Preliminaries). If $p = \text{const}$, then this equation becomes the classical p-Laplacian.

Apart from interesting theoretical considerations, such equations arise in the applied sciences, for instance in fluid dynamics, see, e.g., Diening and Růžička [25], in the study of image processing, see, for example, Chen et al. [22] and electro-rheological fluids, see, e.g., Acerbi and Mingione [1,2]; we also refer to Harjulehto et al. [35] for a recent survey and further references. In spite of the symbolic similarity to the constant exponent p-harmonic equation, various unexpected phenomena may occur when the exponent is a function, for instance the minimum of the $p(\cdot)$-Dirichlet energy may not exist even in the one-dimensional case for smooth functions p; also smooth functions need not be dense in the corresponding variable exponent Sobolev spaces. Although Eq. (1.1) is the Euler-Lagrange equation of the $p(\cdot)$-Dirichlet energy and thus is natural to study, it has many disadvantages comparing to the $p = \text{const}$ case. For instance, solutions of (1.1) are, in general, not scalable, also the Harnack inequality is nonhomogeneous with constant depending on solution. In a consequence, the analysis of nonstandard growth equation is often difficult and leads to technical and nontrivial estimates (nevertheless, see Adamowicz and Hästö [4,5] for a variant of Eq. (1.1) that overcomes some of the aforementioned difficulties, the so-called strong $p(\cdot)$-harmonic equation).

The main goal of this paper is to show the boundary Harnack inequality for $p(\cdot)$-harmonic functions on domains satisfying the ball condition (see Theorem 5.4 below). Let us briefly describe the main ingredients leading to this result, as it requires number of auxiliary lemmas and observations which are interesting per se and can be applied in other studies of variable exponent PDEs.

In Sect. 3, we study oscillations of $p(\cdot)$-harmonic functions close to the boundary of a domain and prove, among other results, variable exponent Carleson estimates on NTA domains, cf. Theorem 3.7. Similar estimates play an important role, for instance in studies of the Laplace operator, in particular in relations between the topological boundary and the Martin boundary of the given domain, also in the p-harmonic analysis (see presentation in Sect. 3 for further details and references). The main tools used in the proof of Theorem 3.7 are Hölder continuity up to the boundary, Harnack's inequality and an argument by Caffarelli et al. [21] which, in our situation, relies on various geometric concepts such as quasihyperbolic geodesics and related chaining arguments, also on characterizations of uniform and NTA domains.

Section 4 is devoted to introducing two types of barrier functions, called Wolanski-type and Bauman-type barrier functions, respectively. In the analysis of PDEs, barrier functions appear, for example, in comparison arguments and in establishing growth conditions for functions, see, e.g., Aikawa et al. [7], Lundström [52], Lundström and Vasilis [54] for the setting of p-harmonic functions. Furthermore, barriers can be applied in the solvability of the Dirichlet problem, especially in studies of regular points, see, e.g., Chapter 6 in Heinonen et al. [38] and Chapter 11 in Björn and Björn [19]. We would like to mention that our results on barriers enhance the existing results in variable exponent setting, see Remark 4.2.

In Sect. 5, we prove our main results, a boundary Harnack inequality and growth estimates for $p(\cdot)$-harmonic functions vanishing on a portion of the boundary of a domain $\Omega \subset \mathbb{R}^n$ satisfying the ball condition. We refer to Sect. 2 for a definition of the ball condition and point out that a domain satisfies the ball condition if and only if its boundary is $C^{1,1}$-regular. Let us now briefly sketch our results. Let $w \in \partial\Omega$, $r > 0$ be small and suppose that p is a bounded Lipschitz continuous variable exponent. Assume that u is a positive $p(\cdot)$-harmonic function in $\Omega \cap B(w, r)$ vanishing continuously on $\partial\Omega \cap B(w, r)$. Then, we prove that

$$\frac{1}{C}\frac{d(x,\partial\Omega)}{r} \le u(x) \le C\frac{d(x,\partial\Omega)}{r} \quad \text{whenever} \quad x \in \Omega \cap B(w, r/\tilde{c}), \tag{1.2}$$

for constants $\tilde{c}$ and C whose values depend on the geometry of Ω, variable exponent p and certain features of u and v, see the statement of Theorem 5.4. Here, $d(x, \partial\Omega)$ denotes the Euclidean distance from x to $\partial\Omega$. Inequality (1.2) says that u vanishes at the same rate as the distance to the boundary when x approaches the boundary.

Suppose that v satisfies the same assumptions as u above. An immediate consequence of (1.2) is then the following boundary Harnack inequality:

$$\frac{1}{C} \le \frac{u(x)}{v(x)} \le C \quad \text{whenever} \quad x \in \Omega \cap B(w, r/\tilde{c}),$$

saying that u and v vanishes at the same rate as x approaches the boundary (see Theorem 5.4 in Sect. 5). Among main tools used in the proof of boundary Harnack estimates, let us mention Lemmas 5.1 and 5.3 where we show the lower and upper estimates for the rate of decay of a $p(\cdot)$-harmonic function close to a boundary of the domain. It turns out that the geometry of the domain affects the number and type of parameters on which the rate of decay depends. Namely, our estimates depend on whether a domain satisfies the interior ball condition or the ball condition in Lemma 5.1 , cf. parts (i) and (ii) of Lemma 5.1. Besides the ball condition, the proof of (1.2) uses the barrier functions derived in Sect. 4, the comparison principle and Harnack's inequality. Our approach extends arguments from Aikawa et al. [7] to the case of variable exponents. We point out that the constants in (1.2), and thus also in the boundary Harnack inequality, depend on u and v. Such a dependence is expected for variable exponent PDEs and difficult to avoid, as, e.g., parameters in the Harnack inequality Lemma 3.1 and the barrier functions depend on solutions as well.

Finally, in Sect. 6, we define and study lower and upper estimates for a $p(\cdot)$-harmonic measure. We also prove a weak doubling property for such measures. In the constant exponent setting, similar results were obtained by Eremenko and Lewis [26], Kilpeläinen and Zhong [43] and Bennewitz and Lewis [17]. For $p = \text{const}$, p-harmonic measures were employed to prove boundary Harnack inequalities, see, e.g., [17], Lewis and Nyström [46] and Lundström and Nyström [53]. The p-harmonic measure, defined as in the aforementioned papers, as well as boundary Harnack inequalities, have played a significant role when studying free boundary problems, see, e.g., Lewis and Nyström [48].

2 Preliminaries

We let $\bar{\Omega}$ and $\partial\Omega$ denote, respectively, the closure and the boundary of the set $\Omega \subset \mathbb{R}^n$, for $n \geq 2$. We define $d(y, \Omega)$ to equal the Euclidean distance from $y \in \mathbb{R}^n$ to Ω, while $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$ and $|x| = \langle x, x \rangle^{1/2}$ is the Euclidean norm of x. Furthermore, by $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, we denote a ball centered at point x with radius $r > 0$, and we let dx denote the n-dimensional Lebesgue measure on $\mathbb{R}^n$. If $\Omega \subset \mathbb{R}^n$ is open and $1 \leq q < \infty$, then by $W^{1,q}(\Omega)$, $W_0^{1,q}(\Omega)$ we denote the standard Sobolev space and the Sobolev space of functions with zero boundary values, respectively. Moreover, let $\Delta(w, r) = B(w, r) \cap \partial\Omega$. By f_A, we denote the integral average of f over a set A.

For background on variable exponent function spaces, we refer to the monograph by Diening et al. [24].

A measurable function $p\colon \Omega \to [1, \infty]$ is called a *variable exponent* and we denote

$$p_A^+ := \operatorname*{ess\,sup}_{x\in A} p(x), \quad p_A^- := \operatorname*{ess\,inf}_{x\in A} p(x), \quad p^+ := p_\Omega^+ \quad \text{and} \quad p^- := p_\Omega^-$$

for $A \subset \Omega$. If $A = \Omega$ or if the underlying domain is fixed, we will often skip the index and set $p_A = p_\Omega = p$.

In this paper, we assume that our variable exponent functions are bounded, i.e.,

$$1 < p^- \leq p(x) \leq p^+ < \infty \quad \text{for almost every } x \in \Omega.$$

The set of all such exponents in Ω will be denoted $\mathcal{P}(\Omega)$.

The function α defined in a bounded domain Ω is said to be log-*Hölder continuous* if there is constant $L > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{L}{\log(e + 1/|x - y|)}$$

for all $x, y \in \Omega$. We denote $p \in \mathcal{P}^{\log}(\Omega)$ if $1/p$ is log-Hölder continuous, the smallest constant for which $\frac{1}{p}$ is log-Hölder continuous is denoted by $c_{\log}(p)$. If $p \in \mathcal{P}^{\log}(\Omega)$, then

$$|B|^{p_B^+} \approx |B|^{p_B^-} \approx |B|^{p(x)} \approx |B|^{p_B} \tag{2.1}$$

for every ball $B \subset \Omega$ and $x \in B$; here p_B is the harmonic average, $\frac{1}{p_B} := \fint_B \frac{1}{p(x)}\,\mathrm{d}x$. The constants in the equivalences depend on $c_{\log}(p)$ and $\operatorname{diam}\Omega$. One of the immediate consequences of (2.1) is that if $x \in B(w, r)$, then

$$\frac{1}{c} r^{-p(w)} \leq r^{-p(x)} \leq c r^{-p(w)} \tag{2.2}$$

with c depending only on constants in (2.1).

In this paper, we study only log-Hölder continuous or Lipschitz continuous variable exponents. Both types of exponents can be extended to the whole $\mathbb{R}^n$ with their constants unchanged, see [24, Proposition 4.1.7] and McShane-type extension result in Heinonen [37, Theorem 6.2], respectively. Therefore, without loss of generality, we assume below that variable exponents are defined in the whole $\mathbb{R}^n$.

We define a *(semi)modular* on the set of measurable functions by setting

$$\varrho_{L^{p(\cdot)}(\Omega)}(u) := \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x;$$

here, we use the convention $t^\infty = \infty\chi_{(1,\infty]}(t)$ in order to get a left continuous modular, see [24, Chapter 2] for details. The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular $\varrho_{L^{p(\cdot)}(\Omega)}(u/\mu)$ is finite for some $\mu > 0$. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\mu > 0 :\ \varrho_{L^{p(\cdot)}(\Omega)}\left(\tfrac{u}{\mu}\right) \leq 1\right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For a constant function p, it coincides with the standard Lebesgue space. Often, it is assumed that p is bounded, since this condition is known to imply many desirable features for $L^{p(\cdot)}(\Omega)$.

There is not functional relationship between norm and modular, but we do have the following useful inequality:

$$\begin{aligned}\min\left\{\varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\} &\leq \|f\|_{L^{p(\cdot)}(\Omega)} \\ &\leq \max\left\{\varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}}, \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}\right\}.\end{aligned} \tag{2.3}$$

One of the consequences of these relations is the so-called *unit ball property*:

$$\begin{aligned}&\varrho_{L^{p(\cdot)}(\Omega)}(f) \leq 1 \ \Rightarrow\ \|f\|_{L^{p(\cdot)}(\Omega)} \leq 1 \quad \text{and} \\ &\varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^-}} \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq \varrho_{L^{p(\cdot)}(\Omega)}(f)^{\frac{1}{p^+}}.\end{aligned} \tag{2.4}$$

If E is a measurable set of finite measure and p and q are variable exponents satisfying $q \leq p$, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$. In particular, every function $u \in L^{p(\cdot)}(\Omega)$ also belongs to $L^{p^-_\Omega}(\Omega)$. The variable exponent Hölder inequality takes the form

$$\int_\Omega fg\,\mathrm{d}x \leq 2\,\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)}, \tag{2.5}$$

where p' is the point-wise *conjugate exponent*, $1/p(x) + 1/p'(x) \equiv 1$.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In general, smooth functions are not dense in the variable exponent Sobolev space, see Zhikov [61] but the log-Hölder condition suffices to guarantee that they are, see Diening et al. [24, Section 8.1]. In this case, we define *the Sobolev space with zero boundary values*, $W_0^{1,p(\cdot)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The Sobolev conjugate exponent is also defined point-wise, $p^*(x) := \frac{np(x)}{n-p(x)}$ for $p^+ < n$. If p is log-Hölder continuous, the Sobolev–Poincaré inequality

$$\|u - u_\Omega\|_{L^{p^*(\cdot)}(\Omega)} \leq c\,\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \tag{2.6}$$

holds when Ω is a nice domain, for instance convex or John [24, Section 7.2]. If $u \in W_0^{1,p(\cdot)}(\Omega)$, then the inequality $\|u\|_{L^{p^*(\cdot)}(\Omega)} \leq c\,\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ holds in any open set Ω.

Definition 2.1 The *Sobolev $p(\cdot)$-capacity* of a set $\Omega \subset \mathbb{R}^n$ is defined as

$$C_{p(\cdot)}(\Omega) := \inf_u \int_{\mathbb{R}^n} (|u|^{p(x)} + |\nabla u|^{p(x)})\,\mathrm{d}x,$$

where the infimum is taken over all $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$ such that $u \geq 1$ in a neighborhood of Ω.

The properties of $p(\cdot)$-capacity are similar to those in the constant case, see Theorem 10.1.2 in [24]. In particular, $C_{p(\cdot)}$ is an outer measure, see Theorem 10.1.1 in [24].
Another type of capacity used in the paper is the so-called *relative $p(\cdot)$-capacity* which appears for instance in the context of uniform $p(\cdot)$-fatness (see next section and Chapter 10.2 in [24] for more details).

Definition 2.2 The *relative $p(\cdot)$-capacity* of a compact set $K \subset \Omega$ is a number defined by

$$\mathrm{cap}_{p(\cdot)}(K, \Omega) = \inf_u \int_\Omega |\nabla u|^{p(x)}\mathrm{d}x,$$

where the infimum is taken over all $u \in C_0^\infty(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ such that $u \geq 1$ in K.

The definition extends to the setting of general sets in $\mathbb{R}^n$ in the same way as in the case of constant p, cf. [24] for details and further properties of the relative $p(\cdot)$-capacity. In what follows, we will need the following estimate, see Proposition 10.2.10 in [24]: For a bounded log-Hölder continuous variable exponent $p : B(x, 2r) \to (1, n)$, it holds that

$$c(n, p)r^{n-p(x)} \leq \mathrm{cap}_{p(\cdot)}(\overline{B}(x, r), B(x, 2r)). \tag{2.7}$$

The similar upper estimate holds for $r \leq 1$, cf. Lemma 10.2.9 in [24].

Definition 2.3 A function $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is a (sub)solution if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi\,\mathrm{d}x\,(\leq) = 0 \tag{2.8}$$

for all (nonnegative) $\phi \in C_0^\infty(\Omega)$.

In what follows, we will exchangeably be using terms (sub)solution and $p(\cdot)$-(sub)solution. Similarly, we say that u is a *supersolution* (*$p(\cdot)$-supersolution)* if $-u$ is a subsolution. A function which is both a subsolution and a supersolution is called a (weak) solution to the $p(\cdot)$-harmonic equation. A continuous weak solution is called a *$p(\cdot)$-harmonic function*.

Among properties of $p(\cdot)$-harmonic functions, let us mention that they are locally $C^{1,\alpha}$, see, e.g., Acerbi and Mingione [1] or Fan [27, Theorem 1.1]. Another tool, crucial from our point of view, is the comparison principle.

Lemma 2.4 (cf. Lemma 3.5 in Harjulehto et al. [32]) *Let u be a supersolution and v a subsolution such that $u > v$ on $\partial\Omega$ in the Sobolev sense. Then, $u > v$ a.e. in Ω.*

By the standard reasoning, the comparison principle implies the following maximum principle: *If $u \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$ is a $p(\cdot)$-subsolution in Ω, then the maximum of u is attained at the boundary of* Ω. For further discussion on comparison principles in the variable exponent setting, we refer, e.g., to Section 3 in Adamowicz et al. [3].

We close our discussion of basic definitions and results with a presentation of the geometric concepts used in the paper.

Definition 2.5 A domain $\Omega \subset \mathbb{R}^n$ is called a *uniform domain* if there exists a constant $M_\Omega \geq 1$, called a *uniform constant*, such that whenever $x, y \in \Omega$ there is a rectifiable curve $\gamma : [0, l(\gamma)] \to \Omega$, parameterized by arc length, connecting x to y and satisfying the following two conditions:

$$l(\gamma) \leq M_\Omega |x - y|,$$

and

$$\min\{|x - z|, |y - z|\} \leq M_\Omega d(z, \partial\Omega) \quad \text{for each point} \quad z \in \gamma.$$

Definition 2.6 A uniform domain $\Omega \subset \mathbb{R}^n$ with constant M_Ω is called a *nontangentially accessible (NTA) domain* if Ω and its complement $\mathbb{R}^n \setminus \Omega$ satisfy, additionally, the so-called *corkscrew condition*:

For some $r_\Omega > 0$ and for any $w \in \partial\Omega$ and $r \in (0, r_\Omega)$, there exists a point $a_r(w) \in \Omega$ such that

$$\frac{r}{M_\Omega} < |a_r(w) - w| < r \quad \text{and} \quad d\big(a_r(w), \partial\Omega\big) > \frac{r}{M_\Omega}.$$

We note that in fact the (interior) corkscrew condition is implied by a uniform domain, see Bennewitz and Lewis [17] and Gehring [30]. Among examples of NTA domains, we mention quasidisks, bounded Lipschitz domains and domains with fractal boundary such as the von Koch snowflake. A domain with the internal power-type cusp is an example of a uniform domain which fails to be NTA domain. Uniform domains are necessarily John domains, the latter one enclosing, e.g., bounded domains satisfying the interior cone condition. See Näkki and Väisälä [56] and Väisälä [58] for further information on uniform and John domains.

Recall that a *quasihyperbolic distance* k_Ω between points x, y in a domain $\Omega \subsetneq \mathbb{R}^n$ is defined as follows

$$k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{\mathrm{d}s(t)}{\mathrm{d}(\gamma(t), \partial\Omega)}, \tag{2.9}$$

where the infimum is taken over all rectifiable curves γ joining x and y in Ω. Any two points in a uniform domain Ω can always be joined by at least one *quasihyperbolic geodesic*, i.e., a curve for which the above infimum can be achieved. See Bonk et al. [20, Section 2] and Gehring and Osgood [31] for more information.

We end this section by recalling the following geometric definition.

Definition 2.7 A domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the *interior ball condition* with radius $r_i > 0$ if for every $w \in \partial\Omega$ there exists $\eta^i \in \Omega$ such that $B(\eta^i, r_i) \subset \Omega$ and $\partial B(\eta^i, r_i) \cap \partial\Omega = \{w\}$. Similarly, a domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the *exterior ball condition* with radius $r_e > 0$ if for every $w \in \partial\Omega$ there exists $\eta^e \in \mathbb{R}^n \setminus \Omega$ such that $B(\eta^e, r_e) \subset \mathbb{R}^n \setminus \Omega$ and $\partial B(\eta^e, r_e) \cap \partial\Omega = \{w\}$. A domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the *ball condition* with radius r_b if it satisfies both the interior ball condition and the exterior ball conditions with radius r_b.

It is well known that $\Omega \subset \mathbb{R}^n$ satisfies the ball condition if and only if Ω is a $C^{1,1}$-domain. See Aikawa et al. [7, Lemma 2.2] for a proof. We also note that if $\Omega \subset \mathbb{R}^n$ satisfies the ball condition then Ω is a NTA domain and hence also a uniform domain.

Throughout the paper, unless otherwise stated, c and C will denote constants whose values may vary at each occurrence. If c depends on the parameters $a_1, \dots, a_n$, we sometimes write $c(a_1, \dots, a_n)$. When constants depend on the variable exponent $p(\cdot)$, we write "depending on p^-, p^+, $c_{\log}$" in place of "depending on p" whenever dependence on p easily reduces to p^-, p^+, $c_{\log}$.

3 Oscillation and Carleson estimates for $p(\cdot)$-harmonic functions

This section is devoted to discussing some important auxiliary results used throughout the rest of the paper. Namely, in Lemmas 3.4, 3.5 and 3.6, we study oscillations of $p(\cdot)$-harmonic functions over the balls intersecting the boundary of the underlying domain. We also employ geometric concepts such as NTA and uniform domains, quasihyperbolic geodesics and distance together with the Harnack inequality to obtain a supremum estimate for a $p(\cdot)$-harmonic function over a chain of balls. Such estimates, discussed in $p =$ const setting for instance in Aikawa and Shanmugalingam [8] or Holopainen et al. [39], require extra attention for variable exponent $p(\cdot)$ as now constant in the Harnack inequality depends on a $p(\cdot)$-harmonic function and the inequality is nonhomogeneous. In Theorem 3.7, we show the main result of this section, namely the variable exponent Carleson estimate. Such estimates play a crucial role in studies of positive p-harmonic functions, see, e.g., Aikawa and Shanmugalingam [8], also Garofalo [29] for an application of Carleson estimates for a class of parabolic equations. According to our best knowledge, Carleson estimates in the setting of equations with nonstandard growth have not been known so far in the literature. We apply Lemma 3.7 in the studies of $p(\cdot)$-harmonic measures in Sect. 5. Moreover, the geometry of the underlying domain turns out to be important in our investigations, in particular properties of NTA domains and uniform $p(\cdot)$-fatness of the complement come into play.

We begin with recalling the Harnack estimate for $p(\cdot)$-harmonic functions.

Lemma 3.1 (Variable exponent Harnack inequality) *Let p be a bounded log-Hölder continuous variable exponent. Assume that u is a nonnegative $p(\cdot)$-harmonic function in $B(w, 4r)$, for some $w \in \mathbb{R}^n$ and $0 < r < \infty$. Then, there exists a constant c_H, depending on n, p and* $\sup_{\Omega\cap B(w,4r)} u$*, such that*

$$\sup_{B(w,r)} u \le c_H \left(\inf_{B(w,r)} u + r \right).$$

Remark 3.2 The variable Harnack inequality in the above form was proved by Alkhutov [9] (see also Alkhutov and Krasheninnikova [10]) and subsequently improved to embrace the case of unbounded solutions by Harjulehto et al. [36, Theorem 3.9]. There, c_H depends only on n, p and the $L^{q's}(B(w, 4r))$-norm of u for $1 < q < \frac{n}{n-1}$ and $s > p^+_{B(w,4r)} - p^-_{B(w,4r)}$.

In what follows we will often iterate the Harnack inequality, and therefore, we need to carefully estimate the growth of constants involved in such iterations. Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with constant M_Ω (for the definition of uniform domains and related concepts see the discussion in the end of Sect. 2). We follow the argument in the proof of Lemma 3.9 in Holopainen et al. [39] and note that a quasihyperbolic geodesic joining two points in Ω is an M'-uniform curve with M' depending only on M_Ω, cf. discussion in Gehring and Osgood

[31]. Let now x and y be given points in $B(w, \frac{r}{M'}) \cap \Omega$ for $w \in \partial\Omega$ and some fixed $r > 0$. As in [39], we find a sequence of balls B_i, $i = 1, \dots, N$ covering quasihyperbolic geodesic γ joining x and y in Ω (such a geodesic always exists for points in uniform domains, see discussion preceding the proof of [39, Lemma 3.9]) and satisfying the following conditions [recall that $k_\Omega(x, y)$ stands for a quasihyperbolic distance between points x and y and is given in (2.9)]:

1. $B_i \cap B_{i+1} \neq \emptyset$ for each i,
2. $2B_i \subset B(w, 4r) \cap \Omega$,
3. $N \le 3k_\Omega(x, y)$.

We estimate the quasihyperbolic distance $k_\Omega(x, y)$ similarly as in formula (16) in Aikawa and Shanmugalingam [8, Section 4]. Among other facts, we employ the definition of John curve. Assume that $d(x, \partial\Omega) \le d(y, \partial\Omega)$ and note that then for a John curve γ, parametrized by arc length so that $\gamma(0) = x$ and $\gamma(l(\gamma)) = y$, the following is true. For all $z \in \gamma$, we have $M_\Omega d(z, \partial\Omega) \ge l(\gamma_{xz})$, where γ_{xz} is the sub curve from x to z. Using this, we see that

$$\begin{aligned} k_\Omega(x, y) &\le \int_\gamma \frac{\mathrm{d}s(t)}{\mathrm{d}(\gamma(t), \partial\Omega)} \le \int_0^{\frac{1}{2}d(x,\partial\Omega)} \frac{\mathrm{d}s}{\frac{1}{2}d(x, \partial\Omega)} + \int_{\frac{1}{2}d(x,\partial\Omega)}^{l(\gamma)} \frac{\mathrm{d}s}{\mathrm{d}(\gamma(t), \partial\Omega)} \\ &\le 1 + M_\Omega \int_{\frac{1}{2}d(x,\partial\Omega)}^{l(\gamma)} \frac{\mathrm{d}s}{s} = 1 + M_\Omega \log s \,\Big|_{\frac{1}{2}d(x,\partial\Omega)}^{l(\gamma)} \le 1 + M_\Omega \log s \,\Big|_{\frac{1}{2}d(x,\partial\Omega)}^{M_\Omega d(y,\partial\Omega)} \\ &= 1 + M_\Omega^2 + M_\Omega \log 2 + M_\Omega \log\left(\frac{d(y, \partial\Omega)}{d(x, \partial\Omega)}\right). \end{aligned}$$

Combining this with the estimate for the number of balls N, we get

$$N \le 9M_\Omega^2 + 3M_\Omega \log\left(\frac{d(y, \partial\Omega)}{d(x, \partial\Omega)}\right), \tag{3.1}$$

whenever $d(x, \partial\Omega) \le d(y, \partial\Omega)$. This estimate can be used in the iteration of Harnack inequality as follows.

Suppose that $x, y \in B(w, \frac{r}{M'})$. Then, by the variable exponent Harnack inequality (Lemma 3.1) and the construction of the chain of balls B_i above, we have that

$$\begin{aligned} u(x) &\le \sup_{B_1(x,r_1)} u(x) \le c_H \left(\inf_{B_1} u + r_1\right) \le \cdots \le \\ &\le c_H^N u(y) + c_H^N r_1 + c_H^{N-1} r_2 + \cdots + r_N \le c_H^N u(y) + c_H^N N r \\ &\le C^N\big(u(y) + r\big). \end{aligned}$$

By using (3.1), we find that

$$C^N \le C^{9M_\Omega^2 + 3M_\Omega \log\left(\frac{d(y,\partial\Omega)}{d(x,\partial\Omega)}\right)} \le C^{9M_\Omega^2} C^{\log\left(\frac{d(y,\partial\Omega)}{d(x,\partial\Omega)}\right)^{3M_\Omega}} \le C^{9M_\Omega^2}\left(\frac{d(y, \partial\Omega)}{d(x, \partial\Omega)}\right)^{3M_\Omega \log C}, \tag{3.2}$$

whenever $d(x, \partial\Omega) \le d(y, \partial\Omega)$.

In some results of this section, we appeal to notion of *uniform $p(\cdot)$-fatness*. For the sake of completeness of the presentation, we recall necessary definitions, cf. Lukkari [50, Sections 3 and 4] and Holopainen et al. [39, Section 3].

Definition 3.3 We say that Ω has *uniformly $p(\cdot)$-fat complement*, if there exist a radius $r_0 > 0$ and a constant $c_0 > 0$ such that

$$\operatorname{cap}_{p(\cdot)}((\mathbb{R}^n \setminus \Omega) \cap B(x, r), B(x, 2r)) \geq c_0 \operatorname{cap}_{p(\cdot)}(\overline{B}(x, r), B(x, 2r)) \tag{3.3}$$

for all $x \in \mathbb{R}^n \setminus \Omega$ and all $r \leq r_0$.

The next lemma provides an oscillation estimate. Similar result was proven by Lukkari in [50, Proposition 4.2]. However, here, we adapt the discussion from [50] to our case; for instance, we do not require the boundary data to be Hölder continuous.

Lemma 3.4 *Let $\Omega \subset \mathbb{R}^n$ be a domain having a uniformly $p(\cdot)$-fat complement with constants c_0 and r_0. Let further p be a bounded log-Hölder continuous variable exponent satisfying either $p^+ \leq n$ or $p^- > n$. Suppose that $w \in \partial\Omega$, $r > 0$ and u is a $p(\cdot)$-harmonic function in $\Omega \cap B(w, r)$, continuous on $\overline{\Omega} \cap \overline{B}(w, r)$ with $u = 0$ on $\partial\Omega \cap B(w, r)$. Then, there exist β, $0 < \beta \leq 1$, a constant $c > 0$ and a radius $\hat{r}$ such that*

$$\sup_{B(w,\rho)\cap\Omega} u \leq c\left(\frac{\rho}{r}\right)^{\beta} \big(\sup_{B(w,r)\cap\Omega} u + r\big)$$

for all $\rho \leq r/2$ and $r \leq \hat{r}$. The constants β and c depend on n, p, $\sup_{B(w,r)\cap\Omega} u$ and c_0, while $\hat{r}$ depends on n, p^-, p^+, $c_{\log}$ and r_0.

Proof Denote $p_0 := p(w)$ and split the discussion into two cases: $p_0 > n$ and $p_0 \leq n$. We start by proving the lemma for $p_0 > n$. By assumptions, u is continuous on $\overline{B}(w, r) \cap \overline{\Omega}$ with $u = 0$ on $B(w, r) \cap \partial\Omega$. Hence, we may use Theorem 1.2 in Alkhutov and Krasheninnikova [10], with $D = B(w, r) \cap \Omega$ and $f = u$. In a consequence, $f(w) = 0$ and $\operatorname{osc}_{\partial D} f \leq \sup_{B(w,r)\cap\Omega} u$ and we obtain that there exists $c = c(n, p, \sup_{B(w,r)\cap\Omega} u)$ such that

$$\sup_{B(w,\rho)\cap\Omega} u \leq c\left(\frac{\rho}{r}\right)^{1-n/p_0} \sup_{B(w,r)\cap\Omega} u, \tag{3.4}$$

for all $\rho \leq r/4$ and with $r \leq \hat{r}(n, p^+, p^-, c_{\log}, r_0)$. The dependence of $\hat{r}$ on the listed parameters follows from the proof of Theorem 1.2 in [10]. Hence, we conclude the lemma for $p_0 > n$ by taking $\beta = \beta(p_0, n) = 1 - n/p_0$.

Assume now that $p_0 \leq n$. To prove the lemma in this case, we will follow the steps and notation of the proof of Proposition 4.2 in Lukkari [50]. In the applications of Lemma 3.4, we will need to understand the exact dependence on constants, and therefore, we repeat parts of the proof from [50].

Let $\eta \in C_0^\infty(B(w, r))$. Then, $\eta u_+ \in W_0^{1,p(\cdot)}(B(w, r) \cap \Omega)$. Further, $\sup_{B(w,r)\cap\Omega} u_+ = \sup_{B(w,r)\cap\Omega} u$ as u attains the boundary values $u \equiv 0$ continuously on $B(w, r) \cap \partial\Omega$. As in Lukkari's proof, we define $\phi(r) := \sup_{B(w,r)\cap\partial\Omega} u - u(w)$ and $\lambda(r) := \sup_{B(w,r)\cap\partial\Omega} u$ and note that under our assumptions $\phi \equiv \lambda \equiv 0$. Then, we use [50, Formula (3.4)] and [50, Formula (4.2)] which requires $r \leq \hat{r}(n, p^-, p^+)$, cf. Formula (3.2) in [50]. Namely, [50, Formula (3.4)] in our case reads

$$\big(\sup_{B(w,r)\cap\Omega} u + r\big) C^{-1}\gamma(r) \leq \sup_{B(w,r)\cap\Omega} u - \sup_{B(w,r/2)\cap\Omega} u + r. \tag{3.5}$$

The analysis of the proof of [50, Formula (3.4)] and the proof of [50, Theorem 3.3] reveals that

$$\big(\sup_{B(w,r)\cap\Omega} u + r\big)^{p(w)-p(x)} \leq c(c_{\log})\big(\sup_{B(w,r)\cap\Omega} u + 1\big)^{p^+-p^-} := C.$$

The $p(\cdot)$-fatness of the complement of Ω together with the capacity estimate (2.7) imply the following inequalities (cf. [50, Formula (3.5)] and [50, Formula (4.2)]):

$$\begin{aligned}\gamma(r) :&= \left(\frac{\operatorname{cap}_{p(\cdot)}((\mathbb{R}^n \setminus \Omega) \cap B(w, r/2), B(w, r))}{r^{n-p(w)}}\right)^{\frac{1}{p(w)-1}} \\ &\geq \left(\frac{c_0 \operatorname{cap}_{p(\cdot)}(\overline{B}(w, r/2), B(w, r))}{r^{n-p(w)}}\right)^{\frac{1}{p(w)-1}} \\ &\geq (c_0 c(n, p))^{\frac{1}{p(w)-1}}.\end{aligned}$$

Thus, $\gamma_0 := C^{-1}\gamma(r)$ satisfies $c(c_0, n, p, \|u^+\|_{L^\infty(B(w,r)\cap\Omega)}) < \gamma_0 < 1$ and (3.5) reads:

$$\sup_{B(w,r/2)\cap\Omega} u \leq \gamma_1 \left(\sup_{B(w,r)\cap\Omega} u + r \right),$$

where $\gamma_1 := \max\{\gamma_0, 1 - \gamma_0\} < 1$. This inequality is a counterpart of [50, Formula (4.3)]. Note also that $\frac{1}{2} \leq \gamma_1 < 1$. We iterate the above inequality to obtain

$$\sup_{B(w,\frac{r}{2^m})\cap\Omega} u \leq \gamma_1^m \big(\sup_{B(w,r)\cap\Omega} u + c(\gamma_1) r \big),$$

where $c(\gamma_1) < 1$ if $\gamma_1 = \frac{1}{2}$ and $c(\gamma_1) \leq \frac{2\gamma_1^{m+1}}{2\gamma_1 - 1}$ for the remaining values of $\gamma_1 \in (\frac{1}{2}, 1)$. We continue as in [50] to find that for $\beta = \log_2(\frac{1}{\gamma_1})$ it holds

$$\gamma_1^m \leq 2^\beta \left(\frac{\rho}{r}\right)^\beta,$$

where β depends on c_0, n, p and $\sup_{B(w,r)\cap\Omega} u$. Hence, the proof is completed. □

To prove Hölder continuity up to the boundary, we will also use the following oscillation estimate which follows from Theorem 4.2, Lemma 2.8 in Fan and Zhao [28] and Lemma 4.8 in Ladyzhenskaya and Ural'tseva [44]. The careful scrutiny of the presentation in [28] reveals the dependance of c and κ on $\sup_\Omega u$ and structure constants (cf. Lemma 3.5). A similar result is given by Theorem 2.2 in Lukkari [50], but under the assumption that $p^+ \leq n$.

Lemma 3.5 *Let p be a bounded log-Hölder continuous variable exponent and let u be a $p(\cdot)$-harmonic function in Ω and let $B(w, r) \Subset \Omega$. Then, there exist c and κ, $0 < \kappa < 1$, such that for all $0 < \rho \leq r$, it holds that*

$$\operatorname{osc}_{B(w,\rho)} u \leq c \left(\frac{\rho}{r}\right)^\kappa \big(\operatorname{osc}_{B(w,r)} u + r\big).$$

The constants c and κ depend on n, p^+, p^- and $\sup_\Omega u$.

We are now ready to formulate the version of Hölder continuity up to the boundary which will be needed in this paper.

Lemma 3.6 *Let $\Omega \subset \mathbb{R}^n$ be a domain having a uniformly $p(\cdot)$-fat complement with constants c_0 and r_0. Let further p be a bounded log-Hölder continuous variable exponent. Suppose that $w \in \partial\Omega$, $r > 0$ and u is a $p(\cdot)$-harmonic function in $\Omega \cap B(w, 2r)$, continuous on $\overline{\Omega} \cap \overline{B}(w, 2r)$ with $u = 0$ on $\partial\Omega \cap B(w, 2r)$. Let $\gamma = \min\{\kappa, \beta\}$ and $r < \hat{r}$ for β and $\hat{r}$ as in Lemma 3.4 and κ as in Lemma 3.5. Then, there exists $C > 0$ such that*

$$|u(x) - u(y)| \leq C \left(\frac{|x - y|}{r}\right)^\gamma \big(\sup_{B(w,2r)\cap\Omega} u + r \big) \quad \textit{whenever} \quad x, y \in B(w, r) \cap \Omega.$$

Proof Let $x, y \in B(w,r) \cap \Omega$ and let $x_0 \in \partial\Omega$ be such that $d(x, \partial\Omega) = |x - x_0|$. We distinguish two cases.

Case 1. $|x-y| < \frac{1}{2} d(x,\partial\Omega)$. Lemma 3.5 applied with $\rho = |x-y|$ and $r = \tau/2$ for $\tau = d(x,\partial\Omega)$ together with Lemma 3.4 imply the following inequalities:

$$
\begin{aligned}
|u(x)-u(y)| \leq \operatorname{osc}_{B(x,\rho)} u &\leq c\left(\frac{|x-y|}{\tau/2}\right)^{\kappa}\left(\operatorname{osc}_{B(x,\tau/2)} u + \frac{\tau}{2}\right) \\
&\leq c2^{\kappa}\left(\frac{|x-y|}{\tau}\right)^{\kappa}\left(\operatorname{osc}_{B(x_0,\frac{3}{2}\tau)\cap\Omega} u + \frac{\tau}{2}\right) \\
&\leq c2^{\kappa}\left(\frac{|x-y|}{\tau}\right)^{\kappa}\left(\sup_{B(x_0,\frac{3}{2}\tau)\cap\Omega} u + \frac{\tau}{2}\right) \\
&\leq c2^{\kappa}\left(\frac{|x-y|}{\tau}\right)^{\kappa}\left[2^{\beta}\left(\frac{\frac{3}{2}\tau}{2r}\right)^{\beta}\left(\sup_{B(w,2r)\cap\Omega} u + 2r\right) + \frac{\tau}{2}\right] \\
&\leq c3^{\beta}2^{\kappa-\beta}\left(\frac{|x-y|}{\tau}\right)^{\kappa}\left(\frac{\tau}{r}\right)^{\beta}\left(\sup_{B(w,2r)\cap\Omega} u + 2r + r^{\beta}\tau^{1-\beta}\frac{1}{2}\right) \\
&\leq C\left(\frac{|x-y|}{r}\right)^{\kappa}\left(\frac{\tau}{r}\right)^{\beta-\kappa}\left(\sup_{B(w,2r)\cap\Omega} u + 2r\right).
\end{aligned}
$$

If $\beta - \kappa > 0$, then $\left(\frac{\tau}{r}\right)^{\beta-\kappa} < 1$ and we get the assertion for $\gamma = \kappa$. Otherwise, if $\beta - \kappa \leq 0$, then since $|x-y| < \frac{1}{2}\tau$, we have that

$$
\left(\frac{|x-y|}{r}\right)^{\kappa}\left(\frac{\tau}{r}\right)^{\beta-\kappa} < \left(\frac{|x-y|}{r}\right)^{\kappa}\left(\frac{r}{2|x-y|}\right)^{\kappa-\beta} \leq 2^{\beta-\kappa}\left(\frac{|x-y|}{r}\right)^{\beta}.
$$

Thus, the estimate holds for $\gamma = \min\{\kappa, \beta\}$.

Case 2. $|x-y| \geq \frac{1}{2} d(x,\partial\Omega)$. Since $u(x_0) = 0$, we have by Lemma 3.4 that

$$
\begin{aligned}
|u(x)-u(y)| &\leq |u(x)-u(x_0)| + |u(y)-u(x_0)| \\
&\leq 2^{\beta}\left(\frac{|x-x_0|}{r}\right)^{\beta}\left(\sup_{B(x_0,r)\cap\Omega} u + r\right) + 2^{\beta}\left(\frac{|y-x_0|}{r}\right)^{\beta}\left(\sup_{B(x_0,r)\cap\Omega} u + r\right) \\
&\leq 4^{\beta}\left(\frac{|x-y|}{r}\right)^{\beta}\left(\sup_{B(x_0,r)\cap\Omega} u + r\right) \\
&\leq 4^{\beta}\left(\frac{|x-y|}{r}\right)^{\beta}\left(\sup_{B(w,2r)\cap\Omega} u + r\right).
\end{aligned}
$$

Since $|x-y| < r$, the last inequality holds as well with exponent $\gamma = \min\{\kappa, \beta\}$, giving us the assertion of the lemma in this case. The proof of Lemma 3.6 is, therefore, completed. □

Following the proof of Theorem 6.31 in Heinonen et al. [38], one can show that *if the complement of* Ω *satisfies the corkscrew condition at* $w \in \partial\Omega$, *then* $\mathbb{R}^n \setminus \Omega$ *is* $p(\cdot)$*-fat at* w. Indeed, using the elementary properties of the relative $p(\cdot)$-capacity (see Section 10.2 in Dieninig et al. [24], in particular Lemma 10.2.9 in [24] and the discussion following it), one shows that (3.3) holds at w. Here, the log-Hölder continuity of $p(\cdot)$ plays an important role as one also employs property (2.2). Hence, the complement of a NTA domain is uniformly $p(\cdot)$-fat, see Definition 2.6.

We are now in a position to prove the main result of this section, the Carleson-type estimate.

Theorem 3.7 *(Variable exponent Carleson-type estimate) Assume that $\Omega \subset \mathbb{R}^n$ is an NTA domain with constants M_Ω and r_Ω. Let $w \in \partial\Omega$, $0 < r \le r_\Omega$ and p be a bounded log-Hölder continuous variable exponent satisfying either $p^+ \le n$ or $p^- > n$. Suppose that u is a positive $p(\cdot)$-harmonic function in $\Omega \cap B(w, r)$, continuous on $\bar{\Omega} \cap B(w, r)$ with $u = 0$ on $\partial\Omega \cap B(w, r)$. Then, there exist constants c and c' such that*

$$\sup_{\Omega \cap B(w,r')} u \le c\left(u(a_{r'}(w)) + r'\right),$$

where $r' = r/c'$. The constant c depends on n, p, $\sup_{B(w,r)\cap\Omega} u$ and M_Ω while c' depends on n, p^-, p^+, $c_{\log}$ and M_Ω, r_Ω.

Proof We proceed following the main lines of Caffarelli et al. [21]. Let k be a large number to be determined later and assume that

$$k\left(u(a_{r'}(w)) + r'\right) < \sup_{\Omega \cap B(w,r')} u = u(x_1) \tag{3.6}$$

where $x_1 \in \partial B(w, r') \cap \Omega$ by the maximum principle. We want to derive a contradiction if k is chosen large enough.

Suppose first that $d(x_1, \partial\Omega) \ge r'/100$. Since Ω is an NTA domain, it is in particular uniform. Hence, we may assume that r' is so small that any two points in $B(w, 2r') \cap \Omega$ can be connected by a Harnack chain totally contained in $B(w, r) \cap \Omega$. Then, $r' = r/c'$ depends only on M_Ω and r_Ω. Since the L^∞-norm of u is bounded in $B(w, r) \cap \Omega$, we can iterate Harnack's inequality using the same constant for each ball contained in $B(w, r) \cap \Omega$. Thus, the Harnack inequality yields the existence of a constant c_0, which by (3.2) depends only on c_H and M_Ω, and such that

$$u(x_1) \le c_0\left(u(a_{r'}(w)) + r'\right). \tag{3.7}$$

This gives us a contradiction if $k > c_0$, and hence, the proof of Theorem 3.7 follows in the case when $d(x_1, \partial\Omega) \ge r'/100$.

Next, assume that $d(x_1, \partial\Omega) < r'/100$. It follows by the Harnack inequality and discussion before (3.2) that there exist constants $\hat{c}$, $\lambda \in [1, \infty)$, depending only on M_Ω and c_H, such that

$$u(x_1) \le \hat{c}\left(\frac{d(a_{r'}(w), \partial\Omega)}{d(x_1, \partial\Omega)}\right)^{\lambda}\left(u(a_{r'}(w)) + r'\right). \tag{3.8}$$

From (3.6) and (3.8), we see that

$$\frac{d(x_1, \partial\Omega)}{d(a_{r'}(w), \partial\Omega)} < \left(\frac{\hat{c}}{k}\right)^{1/\lambda}. \tag{3.9}$$

Let $x_1^+ \in B(w, r') \cap \partial\Omega$ be a point minimizing $|x_1^+ - x_1|$. By decreasing r' if necessary, we apply Lemma 3.6 for $B(x_1^+, r'/2)$ to obtain

$$u(x_1) - u(x_1^+) = u(x_1) \le C\left(\frac{d(x_1, \partial\Omega)}{r'/4}\right)^{\gamma}\Big(\sup_{B(x_1^+, r'/2)\cap\Omega} u + \frac{r'}{4}\Big), \tag{3.10}$$

where γ and C depend on n, p, $\sup_{B(w,r)\cap\Omega} u$ and M_Ω. The constant c' now depends on n, p^-, p^+, $c_{\log}$, M_Ω and r_Ω. By using the Harnack inequality, the maximum principle, assumption (3.6) together with (3.9) and (3.10) we obtain, for some $x_2 \in \partial B(x_1^+, r'/2) \cap \Omega$, the existence of $\check{c} = \check{c}(c_H, M_\Omega)$ such that

$$\begin{aligned} k\check{c}^{-1}\left(u(a_{\frac{r'}{2}}(x_1^+)) + r'\right) &\le k\check{c}^{-1}\left(\check{c}\left[u(a_{r'}(w)) + r'\right] + r'\right) = ku(a_{r'}(w)) + kr' + \frac{k}{\check{c}}r' \\ &< k\left(1 + \frac{1}{\check{c}}\right)\left(u(a_{r'}(w)) + r'\right) < \left(1 + \frac{1}{\check{c}}\right)u(x_1) \\ &\le \left(1 + \frac{1}{\check{c}}\right)C\left(4\frac{d(x_1, \partial\Omega)}{r'}\right)^{\gamma}\left(u(x_2) + \frac{r'}{4}\right) \\ &\le \left(1 + \frac{1}{\check{c}}\right)C\left(\frac{\hat{c}}{k}\right)^{\gamma/\lambda}\left(u(x_2) + \frac{r'}{4}\right). \end{aligned} \tag{3.11}$$

In the last inequality, we have also used $d(a_{r'}(w), \partial\Omega) \le r'$. Define constant k_1 such that

$$\check{c}\left(1 + \frac{1}{\check{c}}\right)C\left(\frac{\hat{c}}{k_1}\right)^{\gamma/\lambda} = 1.$$

By demanding $k > \max\{c_0, k_1\}$, we obtain

$$k\left(u(a_{\frac{r'}{2}}(x_1^+)) + r'\right) \le u(x_2) + \frac{r'}{4} \quad \text{and} \quad u(x_1) < u(x_2) + \frac{r'}{4}.$$

Let $k > 1$. Then, $kr'/2 \ge r'/4$ and the above inequalities take the following form:

$$k\left(u(a_{\frac{r'}{2}}(x_1^+)) + \frac{r'}{2}\right) \le u(x_2) \quad \text{and} \quad u(x_1) < u(x_2) + \frac{r'}{4}. \tag{3.12}$$

We will now repeat the above argument starting from (3.6) with (3.12) replacing (3.6). As now the initial condition has an additional term on the right-hand side, we provide details of the reasoning. Once those are explained, it will become more apparent how to continue with the recurrence argument. Suppose first that $d(x_2, \partial\Omega) \ge r'/200$. Then, similarly as for x_1 we get from (3.12) and the Harnack inequality that

$$k\left(u(a_{\frac{r'}{2}}(x_1^+)) + \frac{r'}{2}\right) \le u(x_2) \le c_0\left(u(a_{\frac{r'}{2}}(x_1^+)) + \frac{r'}{2}\right),$$

where c_0 is the constant from (3.7). Hence, we again obtain the contradiction if $k > c_0$.

Let now $d(x_2, \partial\Omega) < r'/200$. The discussion similar to that for (3.8) gives us

$$u(x_2) \le \hat{c}\left(\frac{d(a_{\frac{r'}{2}}(x_1^+), \partial\Omega)}{d(x_2, \partial\Omega)}\right)^{\lambda}\left(u(a_{\frac{r'}{2}}(x_1^+)) + \frac{r'}{2}\right). \tag{3.13}$$

From (3.12) and (3.13), we see that

$$\frac{d(x_2, \partial\Omega)}{d(a_{\frac{r'}{2}}(x_1^+), \partial\Omega)} < \left(\frac{\hat{c}}{k}\right)^{1/\lambda}.$$

We take point $x_2^+ \in B(x_1^+, \frac{r'}{2}) \cap \partial\Omega$ minimizing $|x_2 - x_2^+|$ and then apply Lemma 3.6 for $B(x_2^+, \frac{r'}{4})$. In a result, we get

$$u(x_2) \le C\left(\frac{d(x_2, \partial\Omega)}{r'/8}\right)^{\gamma}\Big(\sup_{B(x_2^+, \frac{r'}{4}) \cap \Omega} u + \frac{r'}{8}\Big).$$

Following the same reasoning as in (3.11), we obtain, for some $x_3 \in \partial B(x_2^+, \frac{r'}{4}) \cap \Omega$, that

$$\begin{aligned} k\check{c}^{-1}\left(u(a_{\frac{r'}{4}}(x_2^+)) + \frac{r'}{2}\right) &\leq k\check{c}^{-1}\left(\check{c}\left[u(a_{\frac{r'}{2}}(x_1^+)) + \frac{r'}{2}\right] + \frac{r'}{2}\right) \\ &= ku(a_{\frac{r'}{2}}(x_1^+)) + \frac{kr'}{2} + \frac{k}{2\check{c}}r' \\ &< k\left(1+\frac{1}{\check{c}}\right)\left(u(a_{\frac{r'}{2}}(x_1^+)) + \frac{r'}{2}\right) < \left(1+\frac{1}{\check{c}}\right)u(x_2) \\ &\leq \left(1+\frac{1}{\check{c}}\right)C\left(8\frac{d(x_1,\partial\Omega)}{r'}\right)^{\gamma}\left(u(x_3)+\frac{r'}{8}\right) \\ &\leq \left(1+\frac{1}{\check{c}}\right)C\left(\frac{\hat{c}}{k}\right)^{\gamma/\lambda}\left(u(x_3)+\frac{r'}{8}\right). \end{aligned}$$

Since $k > k_1$ and $kr'/4 \geq r'/8$, we arrive at

$$k\left(u(a_{\frac{r'}{4}}(x_2^+)) + \frac{r'}{4}\right) \leq u(x_3) \quad \text{and} \quad u(x_2) < u(x_3) + \frac{r'}{8}.$$

Having established first two steps of the iteration, we now choose points x_m, x_m^+ in the similar way as we found x_1, x_1^+ and x_2, x_2^+ and get that

$$k\left(u(a_{\frac{r'}{2^m}}(x_m^+)) + \frac{r'}{2^m}\right) \leq u(x_{m+1}) \quad \text{and} \quad u(x_m) < u(x_{m+1}) + \frac{r'}{2^{m+1}}.$$

If $m \to \infty$, then $x_m \to y \in \partial\Omega \cap B(w, 2r')$. Since u is assumed continuous on $\overline{\Omega} \cap B(w,r)$ with $u = 0$ on $\partial\Omega \cap B(w,r)$, we obtain that $u(x_m) \to u(y) = 0$. Hence, we conclude that

$$\begin{aligned} k\left(u(a_{r'}(w)) + r'\right) &< u(x_1) < u(x_2) + \frac{r'}{4} < u(x_3) + \frac{r'}{8} + \frac{r'}{4} < \cdots < \\ &< u(x_m) + \frac{r'}{2} \to \frac{r'}{2} \quad \text{for} \quad m \to \infty. \end{aligned}$$

This gives

$$k\left(u(a_{r'}(w)) + r'\right) < \frac{r'}{2}$$

which leads to $k < 1/2$ and results in the contradiction by demanding $k > \max\{1, c_0, k_1\}$. Thus, the proof of Theorem 3.7 is completed. □

4 Constructions of $p(\cdot)$-barriers

Below, we present two types of barrier functions. The first type is based on a work of Wolanski [59]; however, our Lemma 4.1 improves result of [59], see Remark 4.2. We employ Wolanski-type barriers in the upper and lower boundary Harnack estimates, see Sect. 5. The second type of barriers has been inspired by a work of Bauman [16] who uses barriers in studies of a boundary Harnack inequality for uniformly elliptic equations with bounded coefficients. Both approaches have advantages. On one hand, a radius of a ball for which a Wolanski-type barrier exists, depends on less number of parameters then a radius of a corresponding ball for a Bauman-type barrier, but on the other hand, exponents in Wolanski-type barriers depend on larger number of parameters than exponents in Bauman-type barriers, cf. Lemmas 4.1 and 4.3. Therefore, both types of barriers are useful in applications.

4.1 Upper and lower $p(\cdot)$-barriers of Wolanski-type

Lemma 4.1 *Let $y \in \mathbb{R}^n$ and $r > 0$ be fixed and let p be a Lipschitz continuous variable exponent on $\overline{B}(y, 2r)$. Let further $M > 0$ be given and for $x \in B(y, 2r)$ define functions*

$$\hat{u}(x) = \frac{M}{e^{-\mu} - e^{-4\mu}} \left(e^{-\mu} - e^{-\mu \frac{|x-y|^2}{r^2}} \right) \quad \textit{and} \quad \check{u}(x) = \frac{M}{e^{-\mu} - e^{-4\mu}} \left(e^{-\mu \frac{|x-y|^2}{r^2}} - e^{-4\mu} \right).$$

Then, there exist $r_* = r_*(p^-, ||\nabla p||_{L^\infty})$ and $\mu_* = \mu_*(p^+, p^-, n, ||\nabla p||_{L^\infty}, M)$ such that $\hat{u}(x)$ is a $p(\cdot)$-supersolution and $\check{u}(x)$ is a $p(\cdot)$-subsolution in $B(y, 2r) \setminus B(y, r)$ whenever $\mu \geq \mu_*$ and $r \leq r_*$. Furthermore, it holds that

$$\begin{aligned} &\hat{u}(x) = M \text{ on } \partial B(y, 2r) \quad \text{and} \quad \hat{u}(x) = 0 \quad \text{on } \partial B(y, r), \\ &\check{u}(x) = 0 \text{ on } \partial B(y, 2r) \quad \text{and} \quad \check{u}(x) = M \quad \text{on } \partial B(y, r). \end{aligned}$$

Remark 4.2 We would like to point out that the above theorem improves substantially some results on barrier functions in variable exponent setting, see Corollary 4.1 in Wolanski [59]. Namely in [59], the radius r depends also on M, whereas here, we manage to avoid such a dependence [see (4.7) and (4.8) for details]. This plays a role in the proof of Lemma 5.1.

Proof We begin the proof by noting that for any twice differentiable function u, we have

$$\Delta_{p(x)} u = \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right) = \left\langle \nabla \left(|\nabla u|^{p(x)-2} \right), \nabla u \right\rangle + |\nabla u|^{p(x)-2} \Delta u.$$

Now,

$$\begin{aligned} &\left\langle \nabla \left(|\nabla u|^{p(x)-2} \right), \nabla u \right\rangle \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(|\nabla u|^{p(x)-2} \right) \frac{\partial u}{\partial x_i} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\mathrm{e}^{(p(x)-2) \log |\nabla u|} \right) \frac{\partial u}{\partial x_i} \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \Big\{ (p(x) - 2) \log |\nabla u| \Big\} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i} \\ &= \sum_{i=1}^n \left\{ \frac{\partial p}{\partial x_i} \log |\nabla u| + (p(x) - 2) \frac{1}{|\nabla u|} \frac{\partial}{\partial x_i} (|\nabla u|) \right\} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i} \\ &= \sum_{i=1}^n \left\{ \frac{\partial p}{\partial x_i} \log |\nabla u| + (p(x) - 2) \frac{1}{|\nabla u|^2} \sum_{j=1}^n \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \right\} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i} \\ &= \left\{ \langle \nabla p, \nabla u \rangle \log |\nabla u| + (p(x) - 2) \frac{1}{|\nabla u|^2} \Delta_\infty u \right\} |\nabla u|^{p(x)-2}. \end{aligned}$$

Moreover, assuming that $|\nabla u| > 0$, we obtain the following:

$$\Delta_{p(\cdot)} u \leq (\geq)\, 0 \iff \langle \nabla p, \nabla u \rangle \log |\nabla u| + (p(x) - 2) \frac{\Delta_\infty u}{|\nabla u|^2} + \Delta u \leq (\geq)\, 0. \tag{4.1}$$

From (4.1), we see that comparing to the constant p case, we have the extra term involving no second derivatives but the gradient of both u and $p(\cdot)$ instead.

We begin by showing that $\hat{u}$ is a supersolution. We will find μ, A, B and r such that the function

$$\hat{u}(x) = -A \mathrm{e}^{-\mu \left(\frac{|x-y|}{r} \right)^2} + B, \quad \text{where} \quad r < |x - y| < 2r \tag{4.2}$$

has the desired properties. Differentiation of $\hat{u}$ yields

$$
\begin{aligned}
\hat{u}_{x_i} &= \frac{2A\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}(x_i-y_i), \quad |\nabla\hat{u}| = \frac{2A\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}|x-y|,\\
\hat{u}_{x_ix_j} &= \frac{2A\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\left\{\delta_{ij}-\frac{2\mu}{r^2}(x_i-y_i)(x_j-y_j)\right\}.
\end{aligned}
\tag{4.3}
$$

Next, we observe that

$$
\begin{aligned}
\Delta\hat{u} &= \sum_{i=1}^{n}\hat{u}_{x_ix_i} = \frac{2A\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\sum_{i=1}^{n}\left\{1-\frac{2\mu}{r^2}(x_i-y_i)^2\right\}\\
&= \frac{2A\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\left\{n-\frac{2\mu}{r^2}|x-y|^2\right\},
\end{aligned}
\tag{4.4}
$$

and since $\sum_{i,j=1}^{n}\delta_{ij}(x_i-y_i)(x_j-y_j) = |x-y|^2$ and $\sum_{i,j=1}^{n}(x_i-y_i)^2(x_j-y_j)^2 = |x-y|^4$ we also have

$$
\begin{aligned}
\Delta_\infty\hat{u} &= \sum_{i,j=1}^{n}\hat{u}_{x_ix_j}\hat{u}_{x_i}\hat{u}_{x_j}\\
&= \left(\frac{2A\mu}{r^2}\right)^3\mathrm{e}^{-3\mu\left(\frac{|x-y|}{r}\right)^2}\sum_{i,j=1}^{n}\left\{(x_i-y_i)(x_j-y_j)\delta_{ij}-\frac{2\mu}{r^2}(x_i-y_i)^2(x_j-y_j)^2\right\}\\
&= \left(\frac{2A\mu}{r^2}\right)^3\mathrm{e}^{-3\mu\left(\frac{|x-y|}{r}\right)^2}|x-y|^2\left\{1-\frac{2\mu}{r^2}|x-y|^2\right\}.
\end{aligned}
\tag{4.5}
$$

We collect expressions (4.4) and (4.5) and insert them into (4.1) to obtain the following inequality.

$$
\begin{aligned}
&\langle\nabla p,\nabla\hat{u}\rangle\log|\nabla\hat{u}| + \frac{2A\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\left\{(p(x)-2)\left(1-\frac{2\mu}{r^2}|x-y|^2\right)+n-\frac{2\mu}{r^2}|x-y|^2\right\}\\
&\quad\le 0.
\end{aligned}
$$

We simplify the above condition by using $\langle\nabla p,\nabla u\rangle = \frac{2A\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\langle\nabla p, x-y\rangle$:

$$
\langle\nabla p, x-y\rangle\log|\nabla\hat{u}| - \frac{2\mu}{r^2}|x-y|^2(p(x)-1)+n+p(x)-2\le 0.
$$

This holds true if

$$
2r\|\nabla p\|_{L^\infty}|\log|\nabla\hat{u}|| - 2\mu(p^- -1)+n+p^+-2\le 0.
\tag{4.6}
$$

Next, we demand that our function $\hat{u}$ satisfies $\hat{u}(x) = M$ whenever $x\in\partial B(y,2r)$ and $\hat{u}(x)=0$ whenever $x\in\partial B(y,r)$. These assumptions imply that $A = M/(\mathrm{e}^{-\mu}-\mathrm{e}^{-4\mu})$ and $B = M\mathrm{e}^{-\mu}/(\mathrm{e}^{-\mu}-\mathrm{e}^{-4\mu})$.

We now bound $\log|\nabla\hat{u}|$. By (4.3), we have

$$
|\nabla\hat{u}| = \frac{2M\mu}{\mathrm{e}^{-\mu}-\mathrm{e}^{-4\mu}}\frac{|x-y|}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2},
$$

and hence, upon using $r<|x-y|<2r$, we obtain the following estimate

$$
\frac{2M\mu\mathrm{e}^{-3\mu}}{r(1-\mathrm{e}^{-3\mu})}\le|\nabla\hat{u}|\le\frac{4M\mu}{r(1-\mathrm{e}^{-3\mu})}.
$$

Thus,

$$-3\mu + \log\left(\frac{2M\mu}{r(1-\mathrm{e}^{-3\mu})}\right) \le \log|\nabla\hat{u}| \le \log\left(\frac{4M\mu}{r(1-\mathrm{e}^{-3\mu})}\right).$$

Therefore, we conclude

$$|\log|\nabla\hat{u}|| \le \log\left(\frac{4}{1-\mathrm{e}^{-3\mu}}\right) + |\log M| + |\log r| + \log\mu + 3\mu. \tag{4.7}$$

Assume that μ is large and combine (4.6) together with (4.7) to obtain that $\hat{u}$ is a supersolution provided that the following condition is satisfied.

$$2r\|\nabla p\|_{L^\infty}\left(\log\left(\frac{4}{1-\mathrm{e}^{-3\mu}}\right) + |\log M| + |\log r| + 4\mu\right) - 2\mu(p^- - 1) + n + p^+ - 2 \le 0. \tag{4.8}$$

Upon rearranging terms in (4.8), we obtain the following inequality:

$$\mu\left(8r\|\nabla p\|_{L^\infty} - 2(p^- - 1)\right) + 2r\|\nabla p\|_{L^\infty}\left(\log\left(\frac{4}{1-\mathrm{e}^{-3\mu}}\right) + |\log M| + |\log r|\right) + n + p^+ - 2 \le 0.$$

Pick $r_0 = (p^- - 1)/(4\|\nabla p\|_{L^\infty})$. Then, for $r \le r_0$, the above inequality can be satisfied by a large enough μ (upon including the term $|\log M|$ into the first log-term). Moreover, taking $r_* = \min\{r_0, 1/4\}$ ensures that $r|\log r|$ is an increasing function of r for $r \le r_*$. Thus, we conclude that if $r \le r_*$, then there exists $\mu_* = \mu_*(p^+, p^-, n, ||\nabla p||_{L^\infty}, M)$ such that $\hat{u}$ is a supersolution for $\mu \ge \mu_*$. This completes the proof for the supersolution.

Next, we want to show that $\check{u}$ is a subsolution. We will find μ, C, D and r in the function

$$\check{u}(x) = C\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2} + D, \quad \text{where} \quad r < |x-y| < 2r.$$

In this case,

$$\begin{aligned}
\check{u}_{x_i} &= -\frac{2C\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}(x_i - y_i), \quad |\nabla\check{u}| = \frac{2C\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}|x-y|,\\
\check{u}_{x_ix_j} &= -\frac{2C\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\left\{\delta_{ij} - \frac{2\mu}{r^2}(x_i-y_i)(x_j-y_j)\right\}.
\end{aligned} \tag{4.9}$$

Similarly to computations in (4.4) and (4.5), we observe that

$$\begin{aligned}
\Delta\check{u} &= -\frac{2C\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\left\{n - \frac{2\mu}{r^2}|x-y|^2\right\},\\
\Delta_\infty\check{u} &= -\left(\frac{2C\mu}{r^2}\right)^3\mathrm{e}^{-3\mu\left(\frac{|x-y|}{r}\right)^2}|x-y|^2\left\{1 - \frac{2\mu}{r^2}|x-y|^2\right\},
\end{aligned}$$

and

$$\langle\nabla p, \nabla u\rangle = -\frac{2C\mu}{r^2}\mathrm{e}^{-\mu\left(\frac{|x-y|}{r}\right)^2}\langle\nabla p, x-y\rangle.$$

Collecting the terms, we obtain from (4.1) that the condition for $\check{u}$ to be a subsolution becomes

$$-2r\|\nabla p\|_{L^\infty}|\log|\nabla\check{u}|| + 2\mu(p^- - 1) - n - p^+ + 2 \ge 0. \tag{4.10}$$

This is equivalent to (4.6). Finally, we check that assumptions $\check{u}(x) = 0$ whenever $x \in \partial B(y, 2r)$ and $\check{u}(x) = M$ whenever $x \in \partial B(y, r)$ imply $C = M/(\mathrm{e}^{-\mu} - \mathrm{e}^{-4\mu})$ and $D = M\mathrm{e}^{-4\mu}/(\mathrm{e}^{-\mu} - \mathrm{e}^{-4\mu})$. Let A be as in the definition of supersolution $\hat{u}$, see (4.2) and cf. the discussion following (4.6). Since $C = A$, we obtain that bounds for $\log(|\nabla \check{u}|)$ are identical to the case of supersolution. Therefore, the proof of the lemma is completed. □

4.2 Upper and lower $p(\cdot)$-barriers of Bauman-type

Lemma 4.3 *Let $y \in \mathbb{R}^n$ and $r > 0$ be fixed, and let p be a Lipschitz continuous variable exponent on $\overline{B}(y, 2r)$. Let further $M > 0$ be given and for $x \in B(y, 2r)$ define functions*

$$\hat{u}(x) = \frac{M}{1 - 2^{-\mu}} \left[1 - \left(\frac{r}{|x - y|} \right)^{\mu} \right] \quad \textit{and} \quad \check{u}(x) = -\frac{2^{-\mu} M}{1 - 2^{-\mu}} \left[1 - \left(\frac{2r}{|x - y|} \right)^{\mu} \right].$$

Then, there exist $\mu_* = \mu_*(p^-, n) > 0$ and $r_* = r_*(p^-, n, \|\nabla p\|_{L^\infty}, M)$ such that $\hat{u}(x)$ is a $p(\cdot)$-supersolution and $\check{u}(x)$ is a $p(\cdot)$-subsolution in $B(y, 2r) \setminus B(y, r)$ whenever $\mu \geq \mu_*$ and $r \leq r_*$. Furthermore, it holds that

$$\begin{aligned} \hat{u}(x) &= M \text{ on } \partial B(y, 2r) \quad \text{and} \quad \hat{u}(x) = 0 \quad \text{on } \partial B(y, r), \\ \check{u}(x) &= 0 \text{ on } \partial B(y, 2r) \quad \text{and} \quad \check{u}(x) = M \quad \text{on } \partial B(y, r). \end{aligned}$$

Proof Let us show first that $\hat{u}$ is a supersolution. This will be done by choosing parameters μ, A, B and r in the function

$$\hat{u}(x) = -A \left(\frac{r}{|x - y|} \right)^{\mu} + B, \quad \text{where} \quad r < |x - y| < 2r.$$

Differentiation of $\hat{u}$ yields

$$\begin{aligned} \hat{u}_{x_i} &= A\mu r^{\mu} |x - y|^{-(\mu+2)} (x_i - y_i), \\ \hat{u}_{x_i x_j} &= A\mu r^{\mu} |x - y|^{-(\mu+4)} \Big\{ |x - y|^2 \delta_{ij} - (\mu + 2)(x_i - y_i)(x_j - y_j) \Big\}. \end{aligned}$$

Next, we calculate the following expressions:

$$\begin{aligned} |\nabla \hat{u}| &= A\mu r^{\mu} |x - y|^{-(\mu+1)}, \\ \Delta \hat{u} &= \sum_{i=1}^{n} \hat{u}_{x_i x_i} = A\mu r^{\mu} |x - y|^{-(\mu+4)} \sum_{i=1}^{n} \Big\{ |x - y|^2 \delta_{ii} - (\mu + 2)(x_i - y_i)^2 \Big\} \\ &= A\mu r^{\mu} |x - y|^{-(\mu+4)} \Big\{ n|x - y|^2 - (\mu + 2)|x - y|^2 \Big\} \\ &= A\mu r^{\mu} |x - y|^{-(\mu+2)} \Big\{ n - \mu - 2 \Big\}. \end{aligned} \tag{4.11}$$

As $\sum_{i,j=1}^{n} \delta_{ij}(x_i - y_i)(x_j - y_j) = |x-y|^2$ and $\sum_{i,j=1}^{n}(x_i - y_i)^2(x_j - y_j)^2 = |x-y|^4$ we also get that

$$\begin{aligned}
\Delta_\infty \hat{u} &= \sum_{i,j=1}^{n} \hat{u}_{x_i x_j} \hat{u}_{x_i} \hat{u}_{x_j} \\
&= A^3 \mu^3 r^{3\mu} |x-y|^{-(3\mu+8)} \sum_{i,j=1}^{n} \Big\{ |x-y|^2 \delta_{ij} - (\mu+2)(x_i - y_i)(x_j - y_j) \Big\} \\
&\quad \times (x_i - y_i)(x_j - y_j) \\
&= A^3 \mu^3 r^{3\mu} |x-y|^{-(3\mu+8)} \Big\{ |x-y|^4 - (\mu+2)|x-y|^4 \Big\} \\
&= A^3 \mu^3 r^{3\mu} |x-y|^{-(3\mu+4)} \Big\{ 1 - \mu - 2 \Big\}.
\end{aligned}$$

Clearly, $\hat{u} \in C^2(B(y, 2r) \setminus B(y, r))$ and (4.11) gives us that in the given annulus $|\nabla \hat{u}| > 0$. Recall that by the formal computations $\operatorname{div}\left(|\nabla \hat{u}|^{p(x)-2} \nabla \hat{u}\right) \leq 0$ is equivalent to

$$\langle \nabla p, \nabla \hat{u} \rangle \log |\nabla \hat{u}| + (p(x) - 2) \frac{\Delta_\infty \hat{u}}{|\nabla \hat{u}|^2} + \Delta \hat{u} \leq 0. \tag{4.12}$$

By collecting the above expressions and substituting them into (4.12), we obtain the following inequality:

$$\begin{aligned}
&\langle \nabla p, \nabla \hat{u} \rangle \log |\nabla \hat{u}| + (p(x) - 2) A \mu r^\mu |x-y|^{-(\mu+2)} \big\{ 1 - \mu - 2 \big\} \\
&+ A \mu r^\mu |x-y|^{-(\mu+2)} \big\{ n - \mu - 2 \big\} \leq 0.
\end{aligned} \tag{4.13}$$

Use $\langle \nabla p, \nabla \hat{u} \rangle = A \mu r^\mu |x-y|^{-(\mu+2)} \langle \nabla p, x - y \rangle$ in order to simplify (4.13):

$$\langle \nabla p, x - y \rangle \log |\nabla \hat{u}| - \mu(p(x) - 1) + n - p(x) \leq 0.$$

This holds true if

$$\|\nabla p\|_{L^\infty} |x-y| |\log |\nabla \hat{u}|| - \mu(p^- - 1) + n - p^- \leq 0. \tag{4.14}$$

We now chose $\mu_* = \mu_*(p^-, n)$ so that if $\mu \geq \mu_*$, then we have

$$-\mu(p^- - 1) + n - p^- \leq -1. \tag{4.15}$$

Next, we demand that our function $\hat{u}$ satisfies $\hat{u}(x) = M$ whenever $x \in \partial B(y, 2r)$ and $\hat{u}(x) = 0$ whenever $x \in \partial B(y, r)$. This implies that $A = B = M/(1 - 2^{-\mu})$. Our next step is to find conditions for r so that the first term on the left-hand side of (4.14) does not exceed value 1. Since $|x-y| \leq 2r$ it is enough to ensure that

$$\|\nabla p\|_{L^\infty} |\log |\nabla \hat{u}|| 2r \leq 1. \tag{4.16}$$

Then, the proof will be completed by collecting (4.14), (4.15) and (4.16). Hence, it only remains to satisfy (4.16). We have

$$|\nabla \hat{u}| = \frac{M}{1 - 2^{-\mu}} \mu r^\mu |x-y|^{-(\mu+1)} = \frac{M}{1 - 2^{-\mu}} \mu r^{-1} \left(\frac{r}{|x-y|} \right)^{\mu+1}.$$

Since $r < |x-y| < 2r$ and $\mu > 0$, it holds:

$$\frac{M}{1 - 2^{-\mu}} \frac{\mu}{r} \frac{1}{2^{\mu+1}} \leq |\nabla \hat{u}| \leq \frac{M}{1 - 2^{-\mu}} \frac{\mu}{r}.$$

We choose r so small that the left-hand side is larger than one. Such a requirement leads to condition that $r < r_{**} := \frac{M\mu}{2(2^{\mu}-1)}$ and thus r_{**} depends on M and μ_* and therefore on M, p^- and n. Now, $|\log|\nabla\hat{u}|| \leq |\log\left(M\mu/(1-2^{-\mu})\right) - \log r| \leq |\log\left(M\mu/(1-2^{-\mu})\right)| + |\log r|$. As $\lim_{r\to 0^+} r|\log r| = 0$ we have

$$\|\nabla p\|_{L^\infty}|\log|\nabla\hat{u}||2r \leq \|\nabla p\|_{L^\infty}2r\left\{|\log\left(M\mu/(1-2^{-\mu})\right)| + |\log r|\right\} \leq 1, \tag{4.17}$$

provided that $r \leq r_*$ is small enough. Indeed, if $r|\log r| < 1/(4\|\nabla p\|_{L^\infty})$ and $r < 1/(4\|\nabla p\|_{L^\infty}|\log(2^{\mu+1}r_{**})|)$, then (4.17) holds. In a consequence r_* depends only on M, $\|\nabla p\|_{L^\infty}$, p^- and n. The last inequality completes the proof of (4.16), and hence, we have shown that $\hat{u}$ is a supersolution.

In order to show that $\check{u}$ is a $p(\cdot)$-subsolution, we proceed in the analogous way as in the second part of Lemma 4.1, cf. discussion between formulas (4.9) and (4.10). We define

$$\check{u}(x) = C\left(\frac{2r}{|x-y|}\right)^{\mu} - D \quad \text{where} \quad r < |x-y| < 2r.$$

Similarly to computations for $\hat{u}$, we obtain that with A as in the definition of the supersolution $\hat{u}$

$$C = D = 2^{-\mu}A, \quad \nabla\check{u} = -\nabla\hat{u}, \quad \Delta\check{u} = -\Delta\hat{u}, \quad \text{and} \quad \Delta_\infty\check{u} = -\Delta_\infty\hat{u}. \tag{4.18}$$

Upon collecting these expressions, we use them in (4.1) together with $\operatorname{div}\left(|\nabla\check{u}|^{p(x)-2}\nabla\check{u}\right) \geq 0$. In a consequence, we arrive at the following inequality:

$$\begin{aligned}&\langle\nabla p, \nabla\check{u}\rangle\log|\nabla\check{u}| - (p(x)-2)C\mu(2r)^{\mu}|x-y|^{-(\mu+2)}\left\{1-\mu-2\right\}\\ &-C\mu(2r)^{\mu}|x-y|^{-(\mu+2)}\left\{n-\mu-2\right\} \geq 0.\end{aligned}$$

Using (4.18), we see that the above inequality is the same as (4.13) and also that the bounds for $\log|\nabla\check{u}|$ are the same as in the case of supersolution. Thus, the proof for $p(\cdot)$-subsolutions, and for Lemma 4.3, is completed. □

5 Upper and lower boundary growth estimates: The boundary Harnack inequality

This section contains main result of the paper, namely the proof of the boundary Harnack inequality for positive $p(\cdot)$-harmonic functions on domains satisfying the ball condition, see Theorem 5.4. The proof relies on Lemmas 5.1 and 5.3, where we show the lower and, respectively, the upper estimates for a rate of decay of a $p(\cdot)$-harmonic function close to a boundary of the underlying domain. In particular, Lemmas 5.1 and 5.3 imply stronger result than the usual boundary Harnack inequality, namely that a $p(\cdot)$-harmonic function vanishes at the same rate as the distance function. Moreover, Lemma 5.1 illustrates the following phenomenon: the geometry of the domain effects the sets of parameters on which the rate of decay depends. Indeed, it turns out that constants in our lower estimate depend whether domain satisfies the interior ball condition or the ball condition, cf. parts (i) and (ii) of Lemma 5.1. As a corollary, we also obtain a decay estimate for supersolutions (a counterpart of Proposition 6.1 in Aikawa et al. [7]).

For $w \in \partial\Omega$, we denote by $A_r(w)$ a point satisfying $d(A_r(w), \partial\Omega) = r$ and $|A_r(w)-w| = r$. Existence of such a point is guaranteed by the interior ball condition (with radius r_i) for $r \leq r_i/2$. Recall also that by c_H, we denote the constant from the Harnack inequality, Lemma 3.1.

Lemma 5.1 (Lower estimates) *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the interior ball condition with radius r_i, $w \in \partial\Omega$ and $0 < r < r_i$. Let p be a bounded Lipschitz continuous variable exponent. Assume that u is a positive $p(\cdot)$-harmonic function in $\Omega \cap B(w,r)$ satisfying $u = 0$ on $\partial\Omega \cap B(w,r)$. Then, the following is true.*

(i) There exist constants c and $\tilde{c}$ such that if $\tilde{r} := r/\tilde{c}$ then

$$c\,u(x) \geq \frac{d(x,\partial\Omega)}{r} \quad \textit{for} \quad x \in \Omega \cap B(w,\tilde{r}).$$

The constant $\tilde{c}$ depends only on r_i and p^-, $\|\nabla p\|_{L^\infty}$, while c depends on $\inf_{\Gamma_{w,\tilde{r}}} u$, r_i and $p^+, p^-, n, \|\nabla p\|_{L^\infty}$, where $\Gamma_{w,\tilde{r}} = \{x \in \Omega | \tilde{r} < d(x,\partial\Omega) < 3\tilde{r}\} \cap B(w,r)$. Moreover, c is decreasing in $\inf_{\Gamma_{w,\tilde{r}}} u$.

Assume in addition that Ω satisfies the ball condition with radius r_b and that $0 < r < r_b$.

(ii) Then, there exist constants c_L and $\tilde{c}_L$ such that if $\tilde{r} := r/\tilde{c}_L$ then

$$c_L\,u(x) \geq \frac{d(x,\partial\Omega)}{r} \quad \textit{for} \quad x \in \Omega \cap B(w,\tilde{r}).$$

The constant $\tilde{c}_L$ depends only on r_b and $p^-, \|\nabla p\|_{L^\infty}$, while c_L depends on $\sup_{\Omega\cap B(w,r)} u$, $u(A_{2\tilde{r}}(w))$, r_b and $p^+, p^-, n, \|\nabla p\|_{L^\infty}$. Moreover, c_L is decreasing in $u(A_{2\tilde{r}}(w))$ and increasing in $\sup_{\Omega\cap B(w,r)} u$.

Proof To prove (i), we start by applying Lemma 4.1 to obtain r_*, depending only on $\|\nabla p\|_{L^\infty}$, p^-, such that we can construct barriers in an annulus with radius less than r_*. Assume $\tilde{c}$ to be so large that $\tilde{r} \leq \min\{r_*, r/6\}$ and note that so far $\tilde{c} \geq 6$ depends only on $\|\nabla p\|_{L^\infty}$, p^- and r_i.

Let $x \in \Omega \cap B(w,\tilde{r})$ be arbitrary. Then, there exists $\eta \in \partial\Omega$ such that $d(x,\partial\Omega) = |x-\eta|$. By the interior ball condition at η, we find a point η^i such that $B(\eta^i, r_i) \subset \Omega$ and $\eta \in \partial B(\eta^i, r_i)$. Take $\eta^i_{2\tilde{r}} \in [\eta, \eta^i]$ with $d(\eta^i_{2\tilde{r}}, \partial\Omega) = 2\tilde{r}$. Since $\tilde{r} \leq r/6$ we have $B(\eta^i_{2\tilde{r}}, 2\tilde{r}) \subset \Omega \cap B(w,r)$. Thus, u is a positive $p(\cdot)$-harmonic function in $B(\eta^i_{2\tilde{r}}, 2\tilde{r})$. Next we note that $B(\eta^i_{2\tilde{r}}, \tilde{r}) \subset \Gamma_{w,\tilde{r}}$ and since u is continuous

$$u \geq \inf_{\Gamma_{w,\tilde{r}}} u > 0 \quad \text{in} \quad \Gamma_{w,\tilde{r}}. \tag{5.1}$$

Using (5.1) and Lemma 4.1, we construct a subsolution $\check{u}$ in $B(\eta^i_{2\tilde{r}}, 2\tilde{r}) \setminus B(\eta^i_{2\tilde{r}}, \tilde{r})$ with boundary values $\check{u} \equiv \inf_{\Gamma_{w,\tilde{r}}} u \equiv M$ on $\partial B(\eta^i_{2\tilde{r}}, \tilde{r})$ and $\check{u} \equiv 0$ on $\partial B(\eta^i_{2\tilde{r}}, 2\tilde{r})$. Since $\check{u} \leq u$ on $\partial B(\eta^i_{2\tilde{r}}, \tilde{r})$ and $0 = \check{u} \leq u$ on $\partial B(\eta^i_{2\tilde{r}}, 2\tilde{r})$, we obtain that $\check{u} \leq u$ in $B(\eta^i_{2\tilde{r}}, 2\tilde{r}) \setminus B(\eta^i_{2\tilde{r}}, \tilde{r})$ by the comparison principle (Lemma 2.4). By the above discussion $x \in B(\eta^i_{2\tilde{r}}, 2\tilde{r}) \setminus B(\eta^i_{2\tilde{r}}, \tilde{r})$ and the result will follow by showing that $\check{u}$ does not vanish faster than $d(x,\partial\Omega)$ as $x \to \partial\Omega$. In order to show this, we observe that the derivative of $\check{u}$ in a direction normal to $\partial B(\eta^i_{2\tilde{r}}, 2\tilde{r})$ does not vanish. Indeed, using $\mu = \mu_*$ in Lemma 4.1, $\tilde{r} \leq |x - \eta^i_{2\tilde{r}}| \leq 2\tilde{r}$ together with computations for $\nabla\check{u}$ in (4.9) results in the following estimate:

$$\begin{aligned}\left|\left\langle \nabla\check{u}(x), \frac{x-\eta^i_{2\tilde{r}}}{|x-\eta^i_{2\tilde{r}}|}\right\rangle\right| &= \frac{2\mu_* \inf_{\Gamma_{w,\tilde{r}}} u}{\tilde{r}^2} \frac{\mathrm{e}^{-\mu_*\left(\frac{|x-\eta^i_{2\tilde{r}}|}{\tilde{r}}\right)^2}}{\mathrm{e}^{-\mu_*} - \mathrm{e}^{-4\mu_*}} |x - \eta^i_{2\tilde{r}}| \\ &\geq \frac{2\tilde{c}\mu_* \inf_{\Gamma_{w,\tilde{r}}} u}{r} \frac{\mathrm{e}^{-3\mu_*}}{1-\mathrm{e}^{-3\mu_*}} \geq \frac{1}{cr}. \end{aligned} \tag{5.2}$$

Since $\tilde{c}$ depends only on r_i, $\|\nabla p\|_{L^\infty}$, p^-, while μ_* may depend on $\inf_{\Gamma_{w,\tilde{r}}} u$, $\|\nabla p\|_{L^\infty}$, p^-, p^+, n, the constant c depends only on $\inf_{\Gamma_{w,\tilde{r}}} u$, r_i, $\|\nabla p\|_{L^\infty}$, p^-, p^+, n. By decreasing M if necessary, we also see that c is decreasing in $\inf_{\Gamma_{w,\tilde{r}}} u$. Inequality (5.2) completes the proof of (i) in Lemma 5.1.

To prove (ii), assume $\tilde{c}_L$ to be so large that $\tilde{r} = r/\tilde{c}_L \le \min\{r_*, r/6\}$ and note that now $\tilde{c}_L$ depends only on $\|\nabla p\|_{L^\infty}$, p^- and r_b. We proceed as in the former case but with (5.1) replaced by the following claim:

$$u \ge \frac{1}{c_0} \quad \text{on} \quad \overline{B}(\eta^i_{2\tilde{r}}, \tilde{r}), \tag{5.3}$$

where c_0 depends only on c_H, $u(A_{2\tilde{r}}(w))$, $\|\nabla p\|_{L^\infty}$, r_b, p^-, p^+, n, and c_0 is increasing in c_H and decreasing in $u(A_{2\tilde{r}}(w))$. See details below and Fig. 1. In a consequence, we obtain, instead of (5.2), the following inequality:

$$\left|\left\langle \nabla\check{u}(x), \frac{x-\eta^i_{2\tilde{r}}}{|x-\eta^i_{2\tilde{r}}|}\right\rangle\right| = \frac{2\mu_*}{c_0\tilde{r}^2}\frac{\mathrm{e}^{-\mu_*\left(\frac{|x-\eta^i_{2\tilde{r}}|}{\tilde{r}}\right)^2}}{\mathrm{e}^{-\mu_*}-\mathrm{e}^{-4\mu_*}}|x-\eta^i_{2\tilde{r}}| \ge \frac{2\tilde{c}_L\mu_*}{c_0 r}\frac{\mathrm{e}^{-3\mu_*}}{1-\mathrm{e}^{-3\mu_*}} \ge \frac{1}{c_L r}. \tag{5.4}$$

Since $\tilde{c}_L$ depends only on $\|\nabla p\|_{L^\infty}$, r_b, p^- and c_0, μ_* depend only on c_H, $u(A_{2\tilde{r}}(w))$, r_b, $\|\nabla p\|_{L^\infty}$, p^-, p^+, n, c_L depend only on these parameters. Moreover, c_L is decreasing in $u(A_{2\tilde{r}}(w))$ and increasing in c_H (and so increasing in $\sup_{\Omega\cap B(w,r)} u$ through c_H). Inequality (5.4) completes the proof of Lemma 5.1 under the assumed claim (5.3).

To prove claim (5.3), we proceed as follows. By using Harnack's inequality in $B(A_{2\tilde{r}}(w), r_H)$, for some r_H to be chosen later, we obtain (cf. Lemma 3.1)

$$\sup_{B(A_{2\tilde{r}}(w),r_H)} u \le c_H\left(\inf_{B(A_{2\tilde{r}}(w),r_H)} u + r_H\right)$$

and so $\frac{1}{c_H}u(A_{2\tilde{r}}(w)) - r_H \le u(x)$ in $B(A_{2\tilde{r}}(w), r_H)$. Assume r_H so small that $r_H \le \min\{\frac{1}{2c_H}u(A_{2\tilde{r}}(w)), \tilde{r}\}$ and observe that then

$$\frac{1}{2c_H}u(A_{2\tilde{r}}(w)) \le u(x) \quad \text{in} \quad B(A_{2\tilde{r}}(w), r_H).$$

We now use Lemma 4.1 to find a subsolution $\check{u}$ in $B(A_{2\tilde{r}}(w), 2r_H) \setminus B(A_{2\tilde{r}}(w), r_H)$ satisfying $\check{u} \equiv \frac{1}{2c_H}u(A_{2\tilde{r}}(w))$ on $\partial B(A_{2\tilde{r}}(w), r_H)$ and $\check{u} \equiv 0$ on $\partial B(A_{2\tilde{r}}(w), 2r_H)$. The definition of $A_{2\tilde{r}}(w)$ and $r_H \le \tilde{r}$ give us that $B(A_{2\tilde{r}}(w), 2r_H) \subset \Omega \cap B(w, r)$. By the comparison principle, we obtain that $u \ge \check{u}$ in $B(A_{2\tilde{r}}(w), 2r_H) \setminus B(A_{2\tilde{r}}(w), r_H)$. In particular, $\frac{1}{2c_H}u(A_{2\tilde{r}}(w)) \le u$ and

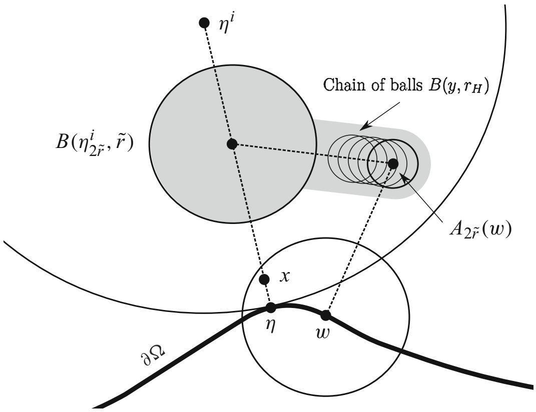
$$\frac{1}{c_1} \le u(x) \quad \text{in} \quad B(A_{2\tilde{r}}(w), 3/2\, r_H).$$

Constant c_1 arises from computing $\check{u}$ for x such that $|x - A_{2\tilde{r}}(w)| = \frac{3}{2}r_H$ (cf. the definition of $\check{u}$ in Lemma 4.1). Furthermore, $c_1 > 2c_H$ depends only on c_H, $u(A_{2\tilde{r}}(w))$, $\|\nabla p\|_{L^\infty}$, p^-, p^+, n, since μ_* in Lemma 4.1 depends only on these parameters.

We proceed by constructing a sequence of barrier functions and building a chain of balls joining points $A_{2\tilde{r}}(w)$ and $\eta^i_{2\tilde{r}}$, where $\eta^i_{2\tilde{r}}$ is the same point as discussed in part (i) of the proof. Using the ball condition, we find that if $\tilde{r}$ is small enough, depending only on r_b, then

$$d([\eta^i_{2\tilde{r}}, A_{2\tilde{r}}(w)], \partial\Omega) \ge \tilde{r}.$$

Fig. 1 The geometry in the proof of Claim (5.3). The chain of annuli $B(y, 2r_H) \setminus B(y, r_H)$ covers the *grey-shaded area*. Using this chain and associated subsolutions from Lemma 4.1, we prove that $u \geq c_0^{-1}$ on $B(\eta^i_{2\tilde{r}}, \tilde{r})$



That such $\tilde{r}$ can be found follows from the argument similar to the one presented in Sections 2 and 3 in Aikawa et al. [7] as Ω is a $C^{1,1}$-domain, and thus, the unit normal is Lipschitz continuous.

Consider the subsolution in $B(y, 2r_H) \setminus B(y, r_H)$ for a $y \in [\eta^i_{2\tilde{r}}, A_{2\tilde{r}}(w)]$ with boundary values $\frac{1}{c_1}$ on $B(y, r_H)$ and 0 on $B(y, 2r_H)$. Put y as close as possible to point $\eta^i_{2\tilde{r}}$ under the restriction that $B(y, r_H) \subset B(A_{2\tilde{r}}(w), 3/2\, r_H)$. By the comparison principle, we then obtain that

$$\frac{1}{c_2} \leq u(x) \quad \text{in} \quad B\big(A_{2\tilde{r}}(w), 3/2\ r_H\big) \cup B\big(y, 3/2\ r_H\big)$$

where $c_2 > c_1 > 2c_H$ depends only on $c_H, u(A_{2\tilde{r}}(w)), \|\nabla p\|_{L^\infty}, p^-, p^+, n$. Proceeding in this way, we obtain a chain of balls centered at points y which, eventually, contain $\eta^i_{2\tilde{r}}$, see Fig. 1. Indeed, each ball adds distance $r_H/2$ to the length of chain, and hence, the number of balls needed to approach $\eta^i_{2\tilde{r}}$ depends only on $\tilde{r}/r_H$, which in turn depends only on $c_H, u(A_{2\tilde{r}}(w)), \|\nabla p\|_{L^\infty}, p^-$ and r_b. We can proceed in the same way to cover $B(\eta^i_{2\tilde{r}}, \tilde{r})$. Hence, we conclude the existence of c_0 in (5.3), depending only on $c_H, u(A_{2\tilde{r}}(w)), \|\nabla p\|_{L^\infty}, r_b, p^-, p^+, n$. From the above construction and from Lemma 4.1, we also see that c_0 is increasing in c_H and decreasing in $u(A_{2\tilde{r}}(w))$. This completes the proof of (5.3) and therefore the proof of Lemma 5.1. □

Denote u^* the lsc-regularization of a supersolution u (see, e.g., Adamowicz et al. [3, Theorem 3.5] and discussion therein).

Corollary 5.2 (cf. Proposition 6.1 in [7]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the interior ball condition for r_i, and let p be a bounded Lipschitz continuous variable exponent such that $p^+ < n$. Furthermore, let $u \geq 0$ be a supersolution in Ω. If there exists a point $w \in \partial\Omega$ such that*

$$\liminf_{\Omega \ni y \to w} \frac{u^*(y)}{d(y, \partial\Omega)} = 0,$$

then $u^* \equiv 0$ in Ω.

Proof Suppose that $u^* \geq 0$ is a lsc-regularization of supersolution u in Ω such that $u^* \not\equiv 0$. Since the Lipschitz continuity assumption on $p(\cdot)$ implies the Dini type condition (see (5.1) in Hästö et al. [34]), we can apply the strong minimum principle (see Theorem 5.3 in [34]) to obtain $u^* > 0$ in Ω. Hence,

$$m := \inf_{w\in\partial\Omega}\left(\inf_{\Gamma_{w,\tilde r}} u^*(x)\right) > 0,$$

where $\Gamma_{w,\tilde r}$ is as in Lemma 5.1 (i). Clearly, the infimum is finite as well. Indeed, by Theorem 3.5 in Adamowicz et al. [3], we have that u^* is a (quasicontinuous) supersolution. Furthermore, Corollary 4.7 in Harjulehto et al. [36] implies that the $p(\cdot)$-capacity of the polar set of u^* is zero, i.e., $C_{p(\cdot)}(\{u^* = \infty\}) = 0$, see Definition 2.1, also [3] and [36] for further discussion. Thus, $m < \infty$.

Since the comparison principle applies to supersolutions, we proceed as in the proof of Lemma 5.1 part (i) to obtain

$$\frac{u^*(y)}{d(y,\partial\Omega)} \geq \frac{1}{cr} > 0 \quad \text{for all } x \text{ close enough to } \partial\Omega,$$

where c depends only on $m, r_i, \|\nabla p\|_{L^\infty}, p^-, p^+, n$. Thus,

$$\liminf_{\Omega\ni y\to w} \frac{u^*(y)}{d(x,\partial\Omega)} > 0$$

for all $w \in \partial\Omega$ and the corollary is proven. □

We now show the upper boundary growth estimates.

Lemma 5.3 (Upper estimates) *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the exterior ball condition with radius r_e, $w \in \partial\Omega$ and $0 < r < r_e$. Let p be a bounded Lipschitz continuous variable exponent. Assume that u is a positive $p(\cdot)$-harmonic function in $\Omega \cap B(w,r)$ satisfying $u = 0$ on $\partial\Omega \cap B(w,r)$. Then, there exist constants c and $\tilde c$ such that if $\tilde r = r/\tilde c$ then*

$$u(x) \leq c\,\frac{d(x,\partial\Omega)}{r} \quad \textit{for} \quad x \in \Omega \cap B(w,\tilde r).$$

The constant $\tilde c$ depends on r_e and p^-, $\|\nabla p\|_{L^\infty}$ while c depends on $\sup_{B(w,4\tilde r)\cap\Omega} u$, r_e and n, p^+, p^-, $\|\nabla p\|_{L^\infty}$. Moreover, c is increasing in $\sup_{B(w,4\tilde r)\cap\Omega} u$.

Proof We first apply Lemma 4.1 to obtain radius r_*, depending only on $\|\nabla p\|_{L^\infty}$ and p^-, such that we can construct barriers in annulus with radius less than r_*. Assume $\tilde c$ to be so large that $\tilde r \leq \min\{r_*, r/5\}$ and note that so far $\tilde c \geq 5$ depends only on $\|\nabla p\|_{L^\infty}$, p^- and r_e. Let $x \in \Omega \cap B(w,\tilde r)$ be arbitrary. Then, there exists $\eta \in \partial\Omega$ such that $d(x,\partial\Omega) = |x-\eta|$. By the exterior ball condition at η, we find a point η^e such that $B(\eta^e, r_e) \subset \mathbb{R}^n \setminus \Omega$ and $\eta \in \partial B(\eta^e, r_e)$. Take $\eta^e_{\tilde r} \in [\eta, \eta^e]$ with $d(\eta^e_{\tilde r}, \partial\Omega) = \tilde r$. Since $\tilde r \leq r/5$ we have $B(\eta^e_{\tilde r}, 2\tilde r) \cap \Omega \subset \Omega \cap B(w,r)$. We now use Lemma 4.1 with $M := \sup_{B(w,4\tilde r)\cap\Omega} u$ to obtain a $p(\cdot)$-supersolution $\hat u$ in the annulus $B(\eta^e_{\tilde r}, 2\tilde r) \setminus B(\eta^e_{\tilde r}, \tilde r)$ satisfying $\hat u \equiv 0$ on $\partial B(\eta^e_{\tilde r}, \tilde r)$ and $\hat u \equiv \sup_{B(w,4\tilde r)\cap\Omega} u$ on $\partial B(\eta^e_{\tilde r}, 2\tilde r)$. By the comparison principle in Lemma 2.4, we obtain $u \leq \hat u$ in $\Omega \cap B(\eta^e_{\tilde r}, 2\tilde r)$ and since x is in this set the result will follow by showing that $\hat u$ vanishes at least as fast as $d(x,\partial\Omega)$ when $x \to \partial\Omega$. Indeed, putting $\mu = \mu^*$, $\tilde r \leq |x - \eta^e_{\tilde r}| \leq 2\tilde r$ together with computations for $\nabla\hat u$ in (4.3) results in the following estimate:

$$\begin{aligned} \left|\left\langle \nabla\hat u(x), \frac{x-\eta^e_{\tilde r}}{|x-\eta^e_{\tilde r}|}\right\rangle\right| &= \frac{2\mu_*\sup_{B(w,4\tilde r)\cap\Omega} u}{\tilde r^2}\,\frac{\mathrm{e}^{-\mu_*\left(\frac{|x-\eta^e_{\tilde r}|}{\tilde r}\right)^2}}{\mathrm{e}^{-\mu_*}-\mathrm{e}^{-4\mu_*}}\,|x-\eta^e_{\tilde r}| \\ &\leq \frac{4\tilde c\mu_*\sup_{B(w,4\tilde r)\cap\Omega} u}{r}\,\frac{1}{1-\mathrm{e}^{-3\mu_*}} \leq \frac{c}{r}. \end{aligned} \tag{5.5}$$

Since μ_* and $\tilde{c}$ bring dependence on $\sup_{B(w,4\tilde{r})\cap\Omega} u$, r_e and $\|\nabla p\|_{L^\infty}$, p^-, p^+, n, we conclude that c depends on the same set of parameters. By increasing M if necessary, we also see that c is increasing in $\sup_{B(w,4\tilde{r})\cap\Omega} u$, and thus, inequality (5.5) completes the proof of Lemma 5.3. □

We are now in a position to state and prove the main result of the paper.

Theorem 5.4 (Boundary Harnack inequality) *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the ball condition with radius r_b. Let $w \in \partial\Omega$, $0 < r < r_b$ and let p be a bounded Lipschitz continuous variable exponent. Assume that u and v are positive $p(\cdot)$-harmonic functions in $\Omega \cap B(w,r)$, satisfying $u = 0 = v$ on $\partial\Omega \cap B(w,r)$. Then, there exist constants c, C and $\tilde{c}$ such that*

$$(i)\quad \frac{1}{c}\frac{d(x,\partial\Omega)}{r} \le u(x) \le c\frac{d(x,\partial\Omega)}{r}, \qquad (ii)\quad \frac{1}{C} \le \frac{u(x)}{v(x)} \le C \quad \textit{for} \quad x \in \Omega \cap B(w, r/\tilde{c}).$$

The constant $\tilde{c}$ depends on r_b and p^-, $||\nabla p||_{L^\infty}$, constant c depends on n, p^+, p^-, $||\nabla p||_{L^\infty}$, $\sup_{B(w,r)\cap\Omega} u$, $u(A_{2\tilde{r}}(w))$ and r_b, while C depends on the same parameters as c and also on $v(A_{2\tilde{r}}(w))$ and $\sup_{B(w,r)\cap\Omega} v$. Moreover, c and C are increasing in $\sup_{\Omega\cap B(w,r)} u$, $\sup_{\Omega\cap B(w,r)} v$ and decreasing in $u(A_{2\tilde{r}}(w))$, $v(A_{2\tilde{r}}(w))$.

Proof (Proof of Theorem 5.4) We observe that $\sup_{B(w,4\tilde{r})\cap\Omega} u$ in Lemma 5.3 can be replaced by $\sup_{B(w,r)\cap\Omega} u$. Then, Lemma 5.1 and Lemma 5.3 immediately imply the assertion of the theorem. □

Remark 5.5 Conclusion (ii) in Theorem 5.4 is sometimes formulated in the following form (also referred to as a boundary Harnack inequality): For all points $x, y \in \Omega \cap B(w, r/\tilde{c})$ it holds that

$$\frac{1}{C^2} \le \frac{u(x)}{v(x)}\frac{v(y)}{u(y)} \le C^2,$$

where $\tilde{c}$ and C are the constants from Theorem 5.4.

Remark 5.6 In contrast to the corresponding result when p is constant, we do not apply the Carleson estimate in Lemma 5.3 to estimate $\sup_{B(w,4\tilde{r})\cap\Omega} u$ by $cu(A_{2\tilde{r}}(w))$, cf. Aikawa et al. [7, Lemma 3.3]. The reason is mainly the weaker form of the Carleson estimate for variable exponent p-Laplacian, namely the inequality in Theorem 3.7 is nonhomogeneous and constant c in that inequality depends on $\sup_{B(w,r)\cap\Omega} u$ as well. In particular, if we assume that Ω satisfies the ball condition with radius r_b, then it follows that Ω is an NTA domain with constants M_Ω and r_Ω, where M_Ω and r_Ω depend only on r_b. Moreover, as in the Theorem 3.7, let us assume also that $p^+ \le n$ or $p^- > n$ and decrease $\tilde{r}$ if necessary. Then, the Carleson estimate (Theorem 3.7), the Harnack inequality (Lemma 3.1) and the assumption $r < r_b$ yield

$$\sup_{\Omega\cap B(w,4\tilde{r})} u \le c\,(u(a_{4\tilde{r}}(w)) + 4\tilde{r}) \le c\,(u(A_{2\tilde{r}}(w)) + 4r_b)\,, \tag{5.6}$$

for some constant $c = c(n, p, \sup_{B(w,r)\cap\Omega} u, r_b)$. However, using (5.6) in (5.5) leads us to an expression where $u(A_{2\tilde{r}}(w))$ cannot be factored out in a convenient way. In fact, the resulting expression on the right-hand side of (5.5) takes a form: $C/r + C_1$. Moreover, the final constant in (5.5) still depends on $\sup_{\Omega\cap B(w,r)} u$ and $u(A_{2\tilde{r}}(w))$ in a nontrivial manner.

Remark 5.7 Let us discuss how our main results correspond to those for constant p. In such a case the Harnack inequality (Lemma 3.1) does not reduce to the usual one, when $p =$ const: the Harnack constant still depends on u and the estimate remains nonhomogeneous. Moreover, the upper and lower estimates (Lemmas 5.1, 5.3) still depend on $u(A_{2\tilde{r}}(w))$ and $\sup_{\Omega\cap B(w,r)} u$.

Nevertheless, one can slightly modify our proofs and retrieve results from [7]. Indeed, we observe that for constant p barriers used in proofs of Lemmas 5.1 and 5.3 depend on p and n only, while r_* does not arise. One may as well replace barriers by fundamental solution for the p-harmonic equation. Moreover, the claim (5.3) improves to $u(A_{2\tilde{r}}(w)) \le cu(x)$ for a constant c depending only on r_b, p and n by trivial application of the stronger Harnack inequality available for constant p (see, e.g., Avelin et al. [12, Lemma 2.1]) This observation, together with using the stronger variant of the Carleson estimate for constant p (see, e.g., [12, Lemma 2.5]) in Lemma 5.3, in a similar way as described in (5.6), implies the Boundary Harnack inequality for constant p in [7].

6 $p(\cdot)$-Harmonic measure

In this section, we study $p(\cdot)$-harmonic measures. In Lemma 6.2, we show the existence of a $p(\cdot)$-harmonic measure, and in Theorem 6.3, we provide our main results of this section: lower and upper growth estimates for such measures. Finally, using these growth estimates and the Carleson estimate (Theorem 3.7), we conclude in Corollary 6.5 a weak doubling property of the $p(\cdot)$-harmonic measure. Let us now explain motivations for our studies.

Harmonic measures were employed to prove a Boundary Harnack inequality in the setting of harmonic functions, see Dahlberg [23] and Jerison and Kenig [40]. When studying boundary behavior of p-harmonic type functions, various versions of generalizations of harmonic measures have been introduced and studied for $p \neq 2$, see, e.g., Llorente et al. [49]. In the case of constant p ($p \neq 2$), Bennewitz and Lewis employed the doubling property of a p-harmonic measure, first proved in Eremenko and Lewis [26], to obtain a Boundary Harnack inequality for p-harmonic functions in the plane, see Bennewitz and Lewis [17]. This result has been generalized to the setting of Aronsson-type equations by Lewis and Nyström [46] and Lundström and Nyström [53]. The p-harmonic measure, defined as in the aforementioned papers, as well as Boundary Harnack inequalities, have played a significant role when studying free boundary problems, see, for example, Lewis and Nyström [48]. The p-harmonic measure was also used to find the optimal Hölder exponent of p-harmonic functions vanishing near the boundary, see Kilpeläinen and Zhong [42] and Lundström [52]. Moreover, a work of Peres and Sheffield [57] provides discussion of connections between p-harmonic measures, defined in a different way though, and tug-of-war games. As for the equations with nonstandard growth, we mention paper by Lukkari et al. [51], where some upper estimates for $p(\cdot)$-harmonic measures were studied in the context of Wolff potentials.

To prove our results concerning $p(\cdot)$-measures, we begin by stating a Caccioppoli-type estimate.

Lemma 6.1 (Caccioppoli-type estimate) *Let $\Omega \subset \mathbb{R}^n$, p be a bounded log-Hölder continuous variable exponent and assume that $\eta \in C_0^\infty(\Omega)$ with $0 \le \eta \le 1$. If u is a $p(\cdot)$-subsolution in Ω, then*

$$\int_\Omega |\nabla u|^{p(x)} \eta^{p^+} dx \le c \int_\Omega |u|^{p(x)} |\nabla \eta|^{p(x)} dx,$$

where $c = c(p^+)$.

Proof The proof goes the same lines as for the $p = \text{const}$ case, namely one uses in (2.8) a test function $\phi = u\eta^{p^+}$, cf. Lemma 5.3 in Harjulehto et al. [33] for the proof of Caccioppoli estimate in the case of slightly modified $p(\cdot)$-Laplace operator $\operatorname{div}(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u)$). □

The following existence lemma is probably known to experts in the variable exponent analysis, but to our best knowledge has not appeared earlier in the literature. Therefore, we include its proof for the readers convenience.

Lemma 6.2 *Assume that $\Omega \subset \mathbb{R}^n$, $w \in \partial\Omega$, $0 < r < \infty$ and let p be a bounded log-Hölder continuous variable exponent. Suppose that u is a positive $p(\cdot)$-harmonic function in $\Omega \cap B(w, 2r)$, continuous on $\bar{\Omega} \cap B(w, 2r)$ with $u \equiv 0$ on $\partial\Omega \cap B(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r)\backslash\Omega$. Then, there exists a unique finite positive Borel measure μ on $\mathbb{R}^n$, with support in $\partial\Omega \cap B(w, r)$, such that whenever $\psi \in C_0^\infty(B(w, r))$ then*

$$\int\limits_{\mathbb{R}^n} \langle |\nabla u|^{p(x)-2}\nabla u, \nabla\psi\rangle dx = -\int\limits_{\mathbb{R}^n} \psi d\mu. \tag{6.1}$$

Proof We first prove that the extended function is a subsolution in $B(w, r)$. To do so, we begin by showing that the extension, denoted by U, belongs to $W^{1,p(\cdot)}_{\text{loc}}(B(w, r))$. It is immediate that U belongs to $L^{p(\cdot)}(B(w, r))$ and that $\nabla U \in L^{p(\cdot)}(B(w, r))$. To conclude that $U \in W^{1,p(\cdot)}_{\text{loc}}(B(w, r))$ it remains to show that $U \in W^{1,p(\cdot)}_{\text{loc}}(B_R)$ for any ball $B_R \Subset B(w, r)$, which in turn boils down to showing that ∇u is the distributional gradient of U in B_R. Indeed, let $\eta \in C_0^\infty(B_R)$ be arbitrary, and let $\phi \in C_0^\infty((\Omega \cap B_R) \cup \operatorname{supp}\eta)$ be such that $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on $\Omega \cap B_R \cap \operatorname{supp}\eta$. Then, $\eta\phi \in C_0^\infty((\Omega \cap B_R) \cup \operatorname{supp}\eta)$. Since ∇u is the distributional gradient of u in $\Omega \cap B(w, r)$ and $u \equiv 0$ in $B_R \setminus \Omega$, we have

$$\begin{aligned} 0 &= \int_{(\Omega\cap B_R)\cup \operatorname{supp}\eta} (u\nabla(\eta\phi) + \eta\phi\nabla u)\, \mathrm{d}x = \int_{\Omega\cap B_R} (u\nabla(\eta\phi) + \eta\phi\nabla u)\, \mathrm{d}x \\ &= \int_{B_R} u\eta\nabla\phi\, \mathrm{d}x + \int_{\Omega\cap B_R\cap \operatorname{supp}\eta} \phi(U\nabla\eta + \eta\nabla u)\, \mathrm{d}x. \end{aligned}$$

The first integral in the right-hand side is zero, $U = 0$ in $B_R \setminus \Omega$, and hence, ∇u is the distributional gradient of U in B_R, and $U \in W^{1,p(\cdot)}_{\text{loc}}(B(w, r))$. To this end, for the sake of simplicity of notation, denote $u = U$.

Next, we show that if $\psi \in C_0^\infty(B(w, r))$ and $\psi \geq 0$, then

$$\int\limits_{\mathbb{R}^n} \langle |\nabla u|^{p(x)-2}\nabla u, \nabla\psi\rangle \mathrm{d}x \leq 0. \tag{6.2}$$

To prove (6.2), define

$$\psi_1 := [(\delta + \max\{u - \varepsilon, 0\})^\varepsilon - \delta^\varepsilon]\psi.$$

Assume that $\varepsilon, \delta > 0$ are small enough, so that ψ_1 is an admissible test function for Definition 2.3. Then, we follow the steps of Lemma 2.2 in Lundström and Nyström [53] for an A-harmonic operator $A(x, \nabla u) := |\nabla u(x)|^{p(x)-2}\nabla u(x)$ to obtain that u is a subsolution in $B(w, r)$. Hence, (6.2) is true.

Let $K \subset B(w, r)$ be compact. By relation between the modular and the norm (2.3), we have that

$$\|1\|_{L^{p(\cdot)}(K)} \leq \max\{r^{\frac{n}{p^+}}, r^{\frac{n}{p^-}}\}. \tag{6.3}$$

The variable exponent Hölder inequality (2.5) together with (6.3) give us that for every compact $K \subset B(w,r)$ and every $\psi \in C_0^\infty(K)$ it holds that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \langle |\nabla u|^{p(x)-2}\nabla u, \nabla \psi\rangle \mathrm{d}x \right| \le \int_K |\nabla u|^{p(x)-1}|\nabla \psi|\mathrm{d}x \\
&\quad \le 2\sup_K |\nabla\psi| \; \|1\|_{L^{p(\cdot)}(K)} \| |\nabla u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(K)} \\
&\quad \le C \sup_K |\nabla \psi| \max\{r^{\frac{n}{p^+}}, r^{\frac{n}{p^-}}\} \max\left\{ \left(\int_K |\nabla u|^{p(x)}\mathrm{d}x\right)^{\frac{p^+-1}{p^-}}, \left(\int_K |\nabla u|^{p(x)}\mathrm{d}x\right)^{\frac{p^--1}{p^+}} \right\}.
\end{aligned} \tag{6.4}$$

Take $\eta \in C_0^\infty(B(w,2r))$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B(w,r)$ and $|\nabla \eta| \le \frac{C}{r}$ for some $C > 1$. We apply Lemma 6.1 to get

$$\begin{aligned}
\int_K |\nabla u|^{p(x)}\mathrm{d}x &\le \int_{B(w,r)} |\nabla u|^{p(x)}\mathrm{d}x \le \int_{B(w,2r)} |\nabla u|^{p(x)}\eta^{p^+}\mathrm{d}x \\
&\le c(p^+)\int_{B(w,2r)} |u|^{p(x)}|\nabla \eta|^{p(x)}\mathrm{d}x \\
&\le c(p^+)\big(r^{-p^+} + r^{-p^-}\big)\int_{B(w,2r)} |u|^{p(x)}\mathrm{d}x \\
&\le c(p^+, n)\, r^n \big(r^{-p^+} + r^{-p^-}\big)\big(1 + \sup_{\Omega\cap B(w,2r)} u\big)^{p^+} \le C,
\end{aligned} \tag{6.5}$$

for some constant C. By (6.2), (6.4) and (6.5) it follows that μ, as defined in (6.1), is a nonnegative distribution in $B(w,r)$ and hence also a positive measure in $B(w,r)$. Since u is $p(\cdot)$-harmonic in $\Omega \cap B(w,r)$ and $u \equiv 0$ in $B(w,r)\setminus\bar{\Omega}$, μ has support within $\partial\Omega \cap B(w,r)$. □

The following theorem is the main result of this section. In the constant exponent setting similar results are well known, see for example Eremenko and Lewis [26], Kilpeläinen and Zhong [43] and Lundström and Nyström [53]. Our result in the variable exponent setting extends partially [53]. Indeed, by taking $p = p^+ = p^-$ in Theorem 6.3, we retrieve the corresponding estimates for $p = \text{const}$, cf. Lemma 2.7 in [53].

Theorem 6.3 *Let $\Omega \subset \mathbb{R}^n$ be a domain having a uniformly $p(\cdot)$-fat complement with constants c_0 and r_0. Assume that $w \in \partial\Omega$, $0 < r < r_0$ and that p is a log-Hölder continuous variable exponent in Ω with $1 < p^- \le p(\cdot) \le p^+ < n$. Suppose that u is a positive $p(\cdot)$-harmonic function in $\Omega \cap B(w,r)$, continuous on $\bar{\Omega} \cap B(w,r)$ with $u \equiv 0$ on $\partial\Omega \cap B(w,r)$. Extend u to $B(w,r)$ by defining $u \equiv 0$ on $B(w,r)\setminus\Omega$ and denote this extension by u. Then, there exist constants C and $\bar{c}$ such that the measure μ satisfies*

$$(i) \quad \mu(\partial\Omega \cap B(w,\bar{r}))^{\frac{p^+}{p^-(p^--1)}} \le C\bar{r}^{\frac{n-p^+}{p^--1}} \sup_{B(w,3\bar{r})\cap\Omega} u,$$

where $\bar{r} = r/\bar{c}$. The constant C depends on n, p^-, p^+, while $\bar{c}$ depends on n, p^-, p^+, $c_{\log}$, $\sup_{B(w,r)\cap\Omega} u$ and c_0, r_0.

If in addition $p^- > 2$ then we also have, with $\tilde{r} = r/\tilde{c}$,

$$(ii) \quad \sup_{B(w,\tilde{r})\cap\Omega} u \le c\left(\tilde{r}^{\frac{p^+(p^--n)}{(p^+)^2-p^-}} \mu(\partial\Omega \cap B(w,r))^{\frac{p^-}{(p^+)^2-p^-}} + \tilde{r}\right).$$

The constants c and $\tilde{c}$ depend on n, p^-, p^+, $c_{\log}$, $\sup_{B(w,r)\cap\Omega} u$ and c_0, r_0, while c additionally depends on c_H.

Remark 6.4 The assumption that the complement of Ω is uniformly $p(\cdot)$-fat can be replaced by a growth estimate on the solution u near $\partial\Omega$. In particular, we use the uniform $p(\cdot)$-fatness only to be able to apply the Hölder continuity result (Lemma 3.4) giving (6.6).

Proof The proof relies on ideas of the constant p case, see Eremenko and Lewis [26, Lemma 1] and Kilpeläinen and Zhong [43, Lemma 3.1]. However, the setting of variable exponent PDEs is causing difficulties in a straightforward extension of $p = \text{const}$ arguments, namely the lack of homogeneity of $p(\cdot)$-harmonic equation and the fact that the homogeneous Sobolev–Poincaré inequality (2.6) holds for norms but not for modular functions require more caution and delicate approach.

We start by choosing $\bar{c} > 6$ so large that with $\bar{r} = r/\bar{c}$ we obtain

$$\sup_{B(w,3\bar{r})} u < 1 \quad \text{and} \quad \bar{r} < 1. \tag{6.6}$$

That such $\bar{c}$ exists follows by Hölder continuity up to the boundary, that is Lemma 3.4. Indeed, in order to prove (6.6), put $\rho = 3\bar{r} = 3r/\bar{c}$ in Lemma 3.4 to obtain

$$\sup_{B(w,3\bar{r})} u = \sup_{B(w,3\bar{r})\cap\Omega} u \le c\left(\frac{3}{\bar{c}}\right)^{\beta}\big(\sup_{B(w,r)\cap\Omega} u + r\big) \le 1$$

if $\bar{c}$ is large enough. The constant $\bar{c}$ depends on n, p^-, p^+, $c_{\log}$, $\sup_{B(w,r)\cap\Omega} u$ and c_0, r_0.

We now prove the upper bound of the measure in Theorem 6.3. To simplify the notation, we define $\Delta(w,r) := \partial\Omega \cap B(w,r)$. Let $\theta \in C_0^\infty(B(w,2\bar{r}))$ be such that $0 \le \theta \le 1$, $\theta \equiv 1$ on $B(w,\bar{r})$ and $|\nabla\theta| \le \frac{C}{\bar{r}}$ for some $C > 1$. Using Hölder's inequality (2.5) and estimate (6.4), we see that

$$\begin{aligned}\mu(\Delta(w,\bar{r})) &\le \int_{\mathbb{R}^n} \theta d\mu \le \int_{\mathbb{R}^n} |\nabla\theta||\nabla u|^{p(x)-1}\mathrm{d}x \le C\bar{r}^{\frac{n}{p^+}-1}\||\nabla u|^{p(\cdot)-1}\|_{L^{p'(\cdot)}(B(w,2\bar{r}))}\\ &\le C\bar{r}^{\frac{n}{p^+}-1}\max\left\{\left(\int_{B(w,2\bar{r})} |\nabla u|^{p(x)}\mathrm{d}x\right)^{\frac{p^+-1}{p^-}}, \left(\int_{B(w,2\bar{r})} |\nabla u|^{p(x)}\mathrm{d}x\right)^{\frac{p^--1}{p^+}}\right\}.\end{aligned} \tag{6.7}$$

Now, let $\eta \in C_0^\infty(B(w,3\bar{r}))$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B(w,2\bar{r})$ and $|\nabla\eta| \le \frac{C}{\bar{r}}$ for some $C > 1$. We apply Lemma 6.1 and (6.6) to obtain

$$\begin{aligned}\int_{B(w,2\bar{r})} |\nabla u|^{p(x)}\mathrm{d}x &\le \int_{B(w,3\bar{r})} |\nabla u|^{p(x)}\eta^{p^+}\mathrm{d}x \le c(p^+)\int_{B(w,3\bar{r})} |u|^{p(x)}|\nabla\eta|^{p(x)}\mathrm{d}x\\ &\le \frac{C(p^+)}{\bar{r}^{p^+}}\int_{B(w,3\bar{r})} |u|^{p(x)}\mathrm{d}x \le C(p^+,n)\,\bar{r}^{n-p^+}\big(\sup_{B(w,3\bar{r})} u\big)^{p^-}.\end{aligned} \tag{6.8}$$

Using (6.7), (6.8), (6.6) and the assumption $p^+ \le n$, we obtain

$$\mu(\Delta(w,\bar r)) \le C(p^-,p^+,n)\,\bar r^{\frac{n}{p^+}-1}\max\left\{\left(\bar r^{n-p^+}\big(\sup_{B(w,3\bar r)} u\big)^{p^-}\right)^{\frac{p^+-1}{p^-}},\right.$$
$$\left.\left(\bar r^{n-p^+}\big(\sup_{B(w,3\bar r)} u\big)^{p^-}\right)^{\frac{p^--1}{p^+}}\right\}$$
$$\le C(p^-,p^+,n)\,\bar r^{\frac{(n-p^+)p^-}{p^+}}\big(\sup_{B(w,3\bar r)} u\big)^{\frac{p^-}{p^+}(p^--1)}.$$

which completes the proof for upper estimates of the measure μ.

We next prove the lower bound of the measure in Theorem 6.3. To do so let $\tilde r = r/\tilde c$ be a radius, for $\tilde c$ to be determined later, and let h be $p(\cdot)$-harmonic in $B(w,\tilde r)$ with boundary values equal to u on $\partial B(w,\tilde r)$. Note that by assumptions, u is continuous on $\bar\Omega\cap\bar B(w,\tilde r)$, and hence, u is well defined on $\partial B(w,\tilde r)$. Existence of h follows from, e.g., Theorem 3.6 in Adamowicz et al. [3]. By the comparison principle (Lemma 2.4), we see that $0\le u\le h$ in $B(w,\tilde r)$. Now, by the Harnack inequality (Lemma 3.1), we have

$$\sup_{B(w,\tilde r/2)} h \;\le\; c_H\big(\inf_{B(w,\tilde r/2)} h+\tilde r\big)$$

and so

$$\inf_{B(w,\tilde r/2)} h \;\ge\; c_H^{-1}\sup_{B(w,\tilde r/2)} u-\tilde r. \tag{6.9}$$

Using Lemma 3.6, we obtain that for $t<1/4$ we have

$$\sup_{B(w,t\tilde r)} u \le \bar C t^{\gamma}\big(\sup_{B(w,\tilde r/2)} u+\tilde r\big). \tag{6.10}$$

Using (6.9) and (6.10), we see that if $x\in B(w,t\tilde r)$ and t is so small that $\bar C t^{\gamma}\le 1/(2c_H)$, then

$$\begin{aligned} h(x)-u(x) &\ge \inf_{B(w,\tilde r/2)} h-\sup_{B(w,t\tilde r)} u\\ &\ge c_H^{-1}\sup_{B(w,\tilde r/2)} u-\tilde r-\bar C t^{\gamma}\big(\sup_{B(w,\tilde r/2)} u+\tilde r\big)\\ &\ge (2c_H)^{-1}\sup_{B(w,\tilde r/2)} u-\big(1+\bar C t^{\gamma}\big)\tilde r\\ &\ge \beta\sup_{B(w,\tilde r/2)} u-\tilde r=\beta M(\tilde r)-\tilde r. \end{aligned}\tag{6.11}$$

Hence, $M(\tilde r)=\sup_{B(w,\tilde r/2)} u$ while β is a small constant satisfying $\beta\le 1/(2c_H(1+\bar C t^{\gamma}))$ where γ and $\bar C$ are from Lemma 3.6. We note that β depends on n, p^-, p^+, $c_{\log}$, c_H, $\sup_{B(w,r)\cap\Omega} u$ and c_0, r_0. Next, we note that by (6.11), a function

$$\psi := \min_{B(w,\tilde r)}\{h-u,\max\{0,\beta M(\tilde r)-\tilde r\}\} \tag{6.12}$$

is nonnegative in $B(w,\tilde r)$ and belongs to $W_0^{1,p(\cdot)}(B(w,\tilde r))$. Using (6.11), we also see that $\psi=\max\{0,\beta M(\tilde r)-\tilde r\}$ on $B(w,t\tilde r)$.

We will now show that

$$\int_{B(w,\tilde{r})} |\nabla \psi|^{p(x)} \mathrm{d}x \leq 2^{p^+-1} \max\{0, \beta M(\tilde{r}) - \tilde{r}\} \mu(\Delta(w, \tilde{r})). \tag{6.13}$$

To do so, let Γ denote the set of points where $\nabla\psi$ exists and is nonzero and note that

$$\int_{B(w,\tilde{r})} |\nabla \psi|^{p(x)} \mathrm{d}x \leq \int_{\Gamma\cap B(w,\tilde{r})} (|\nabla h| + |\nabla u|)^{p(x)-2} |\nabla h - \nabla u|^2 \mathrm{d}x. \tag{6.14}$$

Moreover, for $\xi, \eta \in \mathbb{R}^n$,

$$\begin{aligned}\langle |\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta, \xi - \eta\rangle &= \frac{1}{2}\left(|\xi|^{p(x)-2} + |\eta|^{p(x)-2}\right)|\xi - \eta|^2 \\ &\quad + \frac{1}{2}\left(|\xi|^{p(x)-2} - |\eta|^{p(x)-2}\right)\left(|\xi|^2 - |\eta|^2\right).\end{aligned}$$

Therefore and since $p^- \geq 2$ by assumption,

$$\begin{aligned}\langle |\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta, \xi - \eta\rangle &\geq \frac{1}{2}\left(|\xi|^{p(x)-2} + |\eta|^{p(x)-2}\right)|\xi - \eta|^2 \\ &\geq \frac{1}{2^{p^+-1}}\left(|\xi| + |\eta|\right)^{p(x)-2}|\xi - \eta|^2.\end{aligned}$$

Upon using the last inequality and the fact that h is $p(\cdot)$-harmonic in $B(w, \tilde{r})$ and ψ is an appropriate test function for h together with Lemma 6.2, we obtain

$$\begin{aligned}&\frac{1}{2^{p^+-1}} \int_{\Gamma\cap B(w,\tilde{r})} (|\nabla h| + |\nabla u|)^{p(x)-2} |\nabla h - \nabla u|^2 \mathrm{d}x \\ &\leq \int_{\Gamma\cap B(w,\tilde{r})} \langle |\nabla h|^{p(x)-2}\nabla h - |\nabla u|^{p(x)-2}\nabla u, \nabla h - \nabla u\rangle \mathrm{d}x \\ &= \int_{B(w,\tilde{r})} \langle |\nabla h|^{p(x)-2}\nabla h, \nabla\psi\rangle \mathrm{d}x - \int_{B(w,\tilde{r})} \langle |\nabla u|^{p(x)-2}\nabla u, \nabla\psi\rangle \mathrm{d}x \\ &= -\int_{B(w,\tilde{r})} \langle |\nabla u|^{p(x)-2}\nabla u, \nabla\psi\rangle \mathrm{d}x = \int_{B(w,\tilde{r})} \psi d\mu \leq \max\{0, \beta M(\tilde{r}) - \tilde{r}\}\mu(B(w, \tilde{r})).\end{aligned}$$

Since measure μ is supported on $\Delta(w, \tilde{r})$, we see that $\mu(B(w, \tilde{r})) = \mu(\Delta(w, \tilde{r}))$. Hence, by the above inequality and (6.14), we see that (6.13) holds true.

Next, by assuming $\tilde{c} \geq \bar{c}$, it follows from (6.6) that we have $\beta M(\tilde{r}) - \tilde{r} < 1$. Note that now $\tilde{c}$ depends on n, p^-, p^+, $c_{\log}$, $\sup_{B(w,r)\cap\Omega} u$ and c_0, r_0. Using this fact and the definition of ψ in (6.12), we get that $0 \leq \psi \leq 1$ and thus $\int_{B(w,\tilde{r})} |\psi|^{p(x)} \mathrm{d}x \leq \omega_n \tilde{r}^n \leq \omega_n$. The classical formula for the volume of the unit ball implies that if $1 \leq n \leq 12$, then $\omega_n > 1$. It follows that

$$\int_{B(w,\tilde{r})} \left|\frac{\psi}{\omega_n^{1/p^-}}\right|^{p(x)} \mathrm{d}x \leq \frac{1}{\omega_n} \int_{B(w,\tilde{r})} |\psi|^{p(x)} \mathrm{d}x \leq 1. \tag{6.15}$$

If $n > 12$, then $\omega_n < 1$ and so in (6.15) instead of ω_n^{1/p^-} one has ω_n^{1/p^+}. Eventually, this effects only the power of ω_n in (6.18) which for $\omega_n < 1$ is $1 - p^+/p^- - p^-/p^+$ instead of

$2 - p^+/p^-$ but has no impact on the other expressions in the discussion below. Therefore, we present the argument only in the case of $\omega_n > 1$.

By the unit ball property (2.4), we get

$$\left\| \left| \frac{\psi}{\omega_n^{1/p^-}} \right| \right\|_{L^{p(\cdot)}(B(w,\tilde{r}))} \leq 1 \quad \text{and} \quad \int\limits_{B(w,\tilde{r})} \left| \frac{\psi}{\omega_n^{1/p^-}} \right|^{p(x)} \mathrm{d}x \leq \left\| \left| \frac{\psi}{\omega_n^{1/p^-}} \right| \right\|^{p^-}_{L^{p(\cdot)}(B(w,\tilde{r}))} \tag{6.16}$$

This estimate, the definition of ψ and the Poincaré–Sobolev type inequality (see Theorem 8.2.4 in Diening et al. [24]), imply the following

$$\begin{aligned}(\beta M(\tilde{r}) - \tilde{r})^{p+}\omega_n (t\tilde{r})^n \omega_n^{-p^+/p^-} &\leq \omega_n^{-p^+/p^-} \int\limits_{B(w,\tilde{r})} |\psi|^{p(x)} \mathrm{d}x \leq \left\| \left| \frac{\psi}{\omega_n^{1/p^-}} \right| \right\|^{p^-}_{L^{p(\cdot)}(B(w,\tilde{r}))} \\ &\leq \frac{C_{Sob}^{p^-}}{\omega_n} \tilde{r}^{p^-} \|\nabla \psi\|^{p^-}_{L^{p(\cdot)}(B(w,\tilde{r}))},\end{aligned} \tag{6.17}$$

where C_{Sob} depends on n and $c_{\log}$. In order to pass from the norm of the gradient to its modular, we use similar approach as in (6.15) and (6.16). For the sake of brevity and clarity of the presentation, we will skip some of the tedious computations.

Without the loss of generality we may assume that $\mu(\Delta(w, \tilde{r})) \leq 2^{1-p^+}$. Indeed, this can be obtained by using the upper bound of $\mu(\Delta(w, \tilde{r}))$ proved above ((i) in Theorem 6.3) together with (6.6) and by decreasing $\tilde{r}$ if necessary. Note that $\tilde{c}$ depends on n, p^-, p^+, $c_{\log}$, $\sup_{B(w,r)\cap\Omega} u$ and c_0, r_0. Then, by (6.13), we have that the modular function of $\nabla\psi$ does not exceed value one and thus, by (2.3)

$$\|\nabla \psi\|^{p^+}_{L^{p(\cdot)}(B(w,\tilde{r}))} \leq \int\limits_{B(w,\tilde{r})} |\nabla \psi|^{p(x)} \mathrm{d}x.$$

We continue estimation in (6.17). Using the above, we arrive at the following inequality

$$(\beta M(\tilde{r}) - \tilde{r})^{p+}\omega_n (t\tilde{r})^n \omega_n^{-p^+/p^-} \leq \frac{C_{Sob}^{p^-}}{\omega_n} \tilde{r}^{p^-} \left(\int\limits_{B(w,\tilde{r})} |\nabla \psi|^{p(x)} \mathrm{d}x \right)^{\frac{p^-}{p^+}}.$$

Hence, upon using (6.13) and including $\omega_n^{2-p^+/p^-}$ into the constant on the right-hand side of the above inequality, we get

$$(\beta M(\tilde{r}) - \tilde{r})^{p^+ - \frac{p^-}{p^+}} t^n \leq C\tilde{r}^{p^- - n} \left(\mu(B(w, \tilde{r}))\right)^{\frac{p^-}{p^+}}, \tag{6.18}$$

for some C depending on n, p^-, p^+ and $c_{\log}$. Recall that according to discussion following (6.10), we have that $\bar{C}t^\gamma \leq 1/(2c_H)$. Choose t such that $\bar{C}t^\gamma = 1/(4c_H)$. Then, (6.18) becomes

$$(\beta M(\tilde{r}) - \tilde{r})^{\frac{(p^+)^2}{p^-} - 1} \leq C\tilde{r}^{p^+ - \frac{p^+}{p^-} n} \mu(B(w, \tilde{r})),$$

for C depending on n, p^-, p^+, $c_{\log}$, c_H, $\sup_{B(w,r)\cap\Omega} u$ and c_0, r_0. Thus, we finally conclude

$$\sup_{B(w,\tilde{r})} u \leq C \left(\tilde{r}^{\frac{p^+(p^- - n)}{(p^+)^2 - p^-}} \mu(B(w, \tilde{r}))^{\frac{p^-}{(p^+)^2 - p^-}} + \tilde{r} \right),$$

for some C as above. Thus, the proof of Theorem 6.3 is completed. □

Using Theorems 3.7 and 6.3, we obtain the following weak doubling property of the $p(\cdot)$-harmonic measure.

Corollary 6.5 *Assume that $\Omega \subset \mathbb{R}^n$ is an NTA domain with constants M_Ω and r_Ω, $w \in \partial\Omega$, $0 < r < r_\Omega$ and let $p(\cdot)$ be a log-Hölder continuous variable exponent in Ω with $2 < p^- \le p(\cdot) \le p^+ < n$. Suppose that u is a positive $p(\cdot)$-harmonic function in $\Omega \cap B(w,r)$, continuous on $\bar{\Omega} \cap B(w,r)$ with $u \equiv 0$ on $\partial\Omega \cap B(w,r)$. Extend u to $B(w,r)$ by defining $u \equiv 0$ on $B(w,r)\backslash\Omega$ and denote this extension by u. Then, the measure μ satisfies the following doubling property:*

$$\mu(\partial\Omega \cap B(w,2s))^{\frac{p^+}{p^-(p^--1)}} \le cs^{\alpha}\left(\mu(\partial\Omega \cap B(w,s))^{\frac{p^-}{(p^+)^2-p^-}} + s^{\beta}\right),$$

where $s = r/c$ and the constant c depends on n, p^-, p^+, $c_{\log}$, c_H, $\sup_{B(w,r)\cap\Omega} u$, M_Ω and r_Ω. The exponents $\alpha = \alpha(n, p^+, p^-)$ and $\beta = \beta(n, p^+, p^-)$ are given by

$$\alpha = \frac{(p^+ - p^-)(p^+(n - p^+ - p^-) + n)}{(p^- - 1)((p^+)^2 - p^-)} \quad \text{and} \quad \beta = \frac{(p^+)^2 - p^- - p^+(p^- - n)}{(p^+)^2 - p^-}.$$

In particular, for $p = p^+ = p^-$, we get $\alpha = 0$ and the term s^β goes away as well. Hence, we retrieve the well-known doubling property of p-harmonic measure when p is constant.

Proof Let c be so large that $2s \le \tilde{r}$ where $\tilde{r}$ is as in Theorem 6.3. Then,

$$\mu(\partial\Omega \cap B(w,2s))^{\frac{p^+}{p^-(p^--1)}} \le cs^{\frac{n-p^+}{p^--1}} \sup_{B(w,6s)\cap\Omega} u.$$

By the variable exponent Carleson estimate (Theorem 3.7) and the Harnack inequality (Lemma 3.1), we have

$$\sup_{B(w,6s)\cap\Omega} u \le c\,(u(a_{6s}(w)) + s) \le c\,(u(a_s(w)) + s) \le c\big(\sup_{B(w,s)\cap\Omega} u + s\big).$$

The result now follows by applying the lower bound of the $p(\cdot)$-harmonic measure in Theorem 6.3 and by simplification of the arising formula. □

References

1. Acerbi, E., Mingione, G.: Regularity results for a class of functionals with nonstandard growth. Arch. Ration. Mech. Anal. **156**(2), 121–140 (2001)
2. Acerbi, E., Mingione, G.: Regularity results for stationary electro-rheological fluids. Arch. Ration. Mech. Anal. **164**(3), 213–259 (2002)
3. Adamowicz, T., Björn, A., Björn, J.: Regularity of $p(\cdot)$-superharmonic functions, the Kellogg property and semiregular boundary points. Ann. Inst. H. Poincaré Anal. Non Lineaire **31**(6), 1131–1153 (2014). doi:10.1016/j.anihpc.2013.07.012
4. Adamowicz, T., Hästö, P.: Mappings of finite distortion and PDE with nonstandard growth. Int. Math. Res. Not. IMRN **10**, 1940–1965 (2010)
5. Adamowicz, T., Hästö, P.: Harnack's inequality and the strong $p(x)$-Laplacian. J. Differ. Equ. **250**(3), 1631–1649 (2011)

6. Aikawa, H.: Boundary Harnack principle and Martin boundary for a uniform domain. J. Math. Soc. Jpn. **53**(1), 119–145 (2001)
7. Aikawa, H., Kilpeläinen, T., Shanmugalingam, N., Zhong, X.: Boundary Harnack principle for p-harmonic functions in smooth euclidean domains. Potential Anal. **26**(3), 281–301 (2007)
8. Aikawa, H., Shanmugalingam, N.: Carleson-type estimates for p-harmonic functions and the conformal Martin boundary of John domains in metric measure spaces. Mich. Math. J. **53**(1), 165–188 (2005)
9. Alkhutov, YuA: The Harnack inequality and the Hölder property of solutions of nonlinear elliptic equations with a nonstandard growth condition. Differ. Equ. **33**(12), 1651–1660 (1997). 1726
10. Alkhutov, YuA, Krasheninnikova, O.V.: Continuity at boundary points of solutions of quasilinear elliptic equations with a nonstandard growth condition, (Russian). Izv. Ross. Akad. Nauk Ser. Mat. **68**(6), 3–60 (2004). English translation in Izv. Math. 68 (2004), no. 6, 1063–1117
11. Ancona, A.: Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien. Ann. Inst. Fourier (Grenoble) **28**(4), 169–213 (1978)
12. Avelin, B., Lundström, N.L.P., Nyström, K.: Boundary estimates for solutions to operators of p-Laplace type with lower order terms. J. Differ. Equ. **250**(1), 264–291 (2011)
13. Avelin, B., Nyström, K.: Estimates for solutions to equations of p-Laplace type in Ahlfors regular NTA-domains. J. Funct. Anal. **266**(9), 5955–6005 (2014)
14. Bañuelos, R., Bass, R., Burdzy, K.: Hölder domains and the boundary Harnack principle. Duke Math. J. **64**(1), 195–200 (1991)
15. Bass, R., Burdzy, K.: A boundary Harnack principle in twisted Hölder domains. Ann. Math. **134**(2), 253–276 (1991)
16. Bauman, P.: Positive solutions of elliptic equations in nondivergence form and their adjoints. Ark. Mat. **22**(2), 153–173 (1984)
17. Bennewitz, B., Lewis, J.L.: On the dimension of p-harmonic measure. Ann. Acad. Sci. Fenn. Math. **30**(2), 459–505 (2005)
18. Bhattacharya, T.: On the properties of ∞-harmonic functions and an application to capacitary convex rings. Electron. J. Differ. Equ. **101**, 1–22 (2002)
19. Björn, A., Björn, J.: Nonlinear potential theory on metric spaces, EMS Tracts in Mathematics, 17. European Mathematical Society, Zurich (2011)
20. Bonk, M., Heinonen, J., Koskela, P.: Uniformizing Gromov hyperbolic spaces. Astérisque, 270, viii+99 pp (2001)
21. Caffarelli, L., Fabes, E., Mortola, S., Salsa, S.: Boundary behavior of nonnegative solutions of elliptic operators in divergence form. Indiana Univ. Math. J. **30**(4), 621–640 (1981)
22. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**(4), 1383–1406 (2006)
23. Dahlberg, B.: On estimates of harmonic measure. Arch. Ration. Mech. Anal. **65**, 275–288 (1977)
24. Diening, L., Harjulehto, P., Hästö, P., Růžička, M.: Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics 2017. Springer, Heidelberg (2011)
25. Diening, L., Růžička, M.: Strong solutions for generalized Newtonian fluids. J. Math. Fluid Mech. **7**, 413–450 (2005)
26. Eremenko, A., Lewis, J.L.: Uniform limits of certain A-harmonic functions with applications to quasiregular mappings. Ann. Acad. Sci. Fenn. Ser. A I Math **16**(2), 361–375 (1991)
27. Fan, X.-L.: Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form. J. Differ. Equ. **235**(2), 397–417 (2007)
28. Fan, X.-L., Zhao, D.: A class of De Giorgi type and Hölder continuity. Nonlinear Anal. **36**(3), 295–318 (1999)
29. Garofalo, N.: Second order parabolic equations in nonvariational forms: boundary Harnack principle and comparison theorems for nonnegative solutions. Ann. Mat. Pura Appl. (4) **138**, 267–296 (1984)
30. Gehring, F.: Uniform domains and the ubiquitous quasidisk. Jahresber. Deutsch. Math. Verein. **89**, 88–103 (1987)
31. Gehring, F.W., Osgood, B.G.: Uniform domains and the quasihyperbolic metric. J. Anal. Math. **36**, 50–74 (1979)
32. Harjulehto, P., Hästö, P., Koskenoja, M., Lukkari, T., Marola, N.: An obstacle problem and superharmonic functions with nonstandard growth. Nonlinear Anal. **67**, 3424–3440 (2007)
33. Harjulehto, P., Hästö, P., Latvala, V.: Minimizers of the variable exponent, non-uniformly convex Dirichlet energy. J. Math. Pures Appl. **89**, 174–197 (2008)
34. Harjulehto, P., Hästö, P., Latvala, V., Toivanen, O.: The strong minimum principle for quasisuperminimizers of non-standard growth. Ann. Inst. H. Poincaré Anal. Non Lineaire **28**(5), 731–742 (2011)
35. Harjulehto, P., Hästö, P., Lê Út, V., Nuortio, M.: Overview of differential equations with non-standard growth. Nonlinear Anal. **72**(12), 4551–4574 (2010)

36. Harjulehto, P., Kinnunen, J., Lukkari, T.: Unbounded supersolutions of nonlinear equations with nonstandard growth. Bound. Value Probl., 20. Art. ID 48348, (2007)
37. Heinonen, J.: Lectures on analysis on metric spaces. Universitext, Springer, New York (2001)
38. Heinonen, J., Kilpeläinen, T., Martio, O.: Nonlinear Potential Theory of Degenerate Elliptic Equations, 2nd edn. Dover, Mineola, New York (2006)
39. Holopainen, I., Shanmugalingam, N., Tyson, J.: On the conformal Martin boundary of domains in metric spaces. Papers on analysis, pp. 147–168, Rep. Univ. Jyväskylä Dep. Math. Stat., vol. 83, Univ. Jyväskylä, Jyväskylä (2001)
40. Jerison, D., Kenig, C.: Boundary behavior of harmonic functions in nontangentially accessible domains. Adv. Math. **46**, 80–147 (1982)
41. Kemper, J.T.: A boundary Harnack principle for Lipschitz domains and the principle of positive singularities. Comm. Pure Appl. Math. **25**, 247–255 (1972)
42. Kilpeläinen, T., Zhong, X.: Removable sets for continuous solutions of quasilinear elliptic equations. Proc. Am. Math. Soc. **130**(6), 1681–1688 (2002)
43. Kilpeläinen, T., Zhong, X.: Growth of entire A-subharmonic functions. Ann. Acad. Sci. Fenn. Math. **28**(1), 181–192 (2003)
44. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Linear and quasilinear elliptic equations translated from the Russian by Scripta Technica, Inc., Translation editor: Leon Ehrenpreis. Academic Press, New York (1968)
45. Lewis, J.L., Nyström, K.: Boundary behaviour for p-harmonic functions in Lipschitz and starlike Lipschitz ring domains. Ann. Sci. École Norm. Sup. (4) **40**(5), 765–813 (2007)
46. Lewis, J., Nyström, K.: The boundary Harnack inequality for infinity harmonic functions in the plane. Proc. Am. Math. Soc. **136**(4), 1311–1323 (2008)
47. Lewis, J.L., Nyström, K.: Boundary behavior and the Martin boundary problem for p-harmonic functions in Lipschitz domains. Ann. Math. (2) **172**(3), 1907–1948 (2010)
48. Lewis, J.L., Nyström, K.: Regularity and free boundary regularity for the p-Laplace operator in Reifenberg flat and Ahlfors regular domains. J. Am. Math. Soc. **25**, 827–862 (2012)
49. Llorente, J., Manfredi, J., Wu, J.-M.: p-Harmonic measure is not additive on null sets. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4**(2), 357–373 (2005)
50. Lukkari, T.: Boundary continuity of solutions to elliptic equations with nonstandard growth. Manus. Math. **132**(3–4), 463–482 (2010)
51. Lukkari, T., Maeda, F.-Y., Marola, N.: Wolff potential estimates for elliptic equations with nonstandard growth and applications. Forum Math. **22**(6), 1061–1087 (2010)
52. Lundström, N.L.P.: Estimates for p-harmonic functions vanishing on a flat. Nonlinear Anal. **74**(18), 6852–6860 (2011)
53. Lundström, N.L.P., Nyström, K.: The boundary Harnack inequality for solutions to equations of Aronsson type in the plane. Ann. Acad. Sci. Fenn. Math. **36**(1), 261–278 (2011)
54. Lundström, N.L.P., Vasilis, J.: Decay of a p-harmonic measure in the plane. Ann. Acad. Sci. Fenn. Math. **38**(1), 351–366 (2013)
55. Nyström, K.: p-Harmonic functions in the Heisenberg group: boundary behaviour in domains well-approximated by non-characteristic hyperplanes. Math. Ann. **357**(1), 307–353 (2013)
56. Näkki, R., Väisälä, J.: John disks. Expo. Math. **9**, 3–43 (1991)
57. Peres, Y., Sheffield, S.: Tug-of-war with noise: a game-theoretic view of the p-Laplacian. Duke Math. J. **145**(1), 91–120 (2008)
58. Väisälä, J.: Uniform domains. Tohoku Math. J. **40**, 101–118 (1988)
59. Wolanski, N.: Local bounds, Harnack inequality and Hölder continuity for divergence type elliptic equations with nonstardard growth. arXiv:1309.2227
60. Wu, J.-M.: Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains. Ann. Inst. Fourier (Grenoble) **28**(4), 147–167 (1978)
61. Zhikov, V.: Density of smooth functions in Sobolev–Orlicz spaces. J. Math. Sci. (N.Y.) **132**(3), 285–294 (2006)